**Instructions**

This final exam is open book. You may use whatever written materials you choose including your class notes and the textbook. You may not use electronic devices; computers, calculators, phones, music players, tablet computers, etc. These permissions should be taken to indicate the limited help that either written material or computational assistance is likely to provide — please do not spend significant time looking up books or computing details. That is not what is being tested here. Marks are awarded for concepts, methods and the expression of ideas.

You should attempt to answer all three questions. They are equally valued.

You have 180 minutes. Please mark your papers with your name and student number.

HINTS:
- Read the questions very carefully.
- Marks are awarded for methods, concepts and expression.
- There are many words in the quiz. So I have marked with a ⋆ in the right margin to indicate where a response is requested of you.
Question 1 — Aircraft pitch dynamics

Figure 1 depicts an aircraft’s longitudinal motion, i.e., without roll or yaw.

We define the following variables associated with the aircraft.

- $x_t$ - the axial velocity of the plane,
- $w_t$ - the vertical velocity of the plane,
- $V$ - the velocity of the plane, which we take fixed in magnitude but not direction,
- $\theta_t$ - the pitch angle of the plane,
- $q_t$ - the pitch angle rate, $\dot{\theta}_t$, of the plane
- $\alpha_t$ - the angle of attack of the plane, which is the angle between the forward velocity vector and the axis of the plane,
- $\delta_t$ - the elevator angle of the two flaps on the tailplane measured clockwise.

In this system, the input signal is the elevator angle, $\delta_t$, and the output signals are $\alpha_t$ and $q_t$. We are ignoring the roll and yaw modes and presume that the elevators are moved together.

The linearized differential equations satisfied by $\alpha_t$ and $q_t$ in this case are

\begin{align*}
\ddot{q}_t + 4\dot{q}_t + 6q_t &= -10\delta_t, \\
\ddot{\alpha}_t + 4\dot{\alpha}_t + 6\alpha_t &= -10\dot{\delta}_t - 12\delta_t.
\end{align*}

Part i: Take the Laplace transform of each of the dynamic equations (1) and (2) above and identify the zero-input response transform and the zero-state response transform of each. Identify the transfer functions from $\delta_t$ to $q_t$ and from $\delta_t$ to $\alpha_t$ and compute the poles and zeros of each of these transfer functions.

Taking Laplace transforms:

$$s^2Q(s) - sq(0^-) - \dot{q}(0^-) + 4sQ(s) - 4q(0^-) + 6Q(s) = -10\Delta(s),$$

$$Q(s) = \frac{sq(0^-) + \dot{q}(0^-) + 4q(0^-)}{s^2 + 4s + 6} + \frac{-10}{s^2 + 4s + 6}\Delta(s).$$

The zero-input response transform is $\frac{sq(0^-) + \dot{q}(0^-) + 4q(0^-)}{s^2 + 4s + 6}$. The zero-state response transform is $\frac{-10}{s^2 + 4s + 6}\Delta(s)$. The transfer function from $\delta_t$ to $q_t$ is $\frac{-10}{s^2 + 4s + 6}$. This has no zeros and poles at $s = -2 \pm j\sqrt{2}$.

$$s^2A(s) - s\alpha(0^-) - \dot{\alpha}(0^-) + 4sA(s) - 4\alpha(0^-) + 6A(s) = -10s\Delta(s) - 12\Delta(s),$$

$$A(s) = \frac{s\alpha(0^-) + \dot{\alpha}(0^-) + 4\alpha(0^-)}{s^2 + 4s + 6} + \frac{-10s - 12}{s^2 + 4s + 6}\Delta(s).$$

Note that we do not consider initial conditions associated with the input; they are taken to be zero, since they are ours to choose. The zero-input response transform is $\frac{s\alpha(0^-) + \dot{\alpha}(0^-) + 4\alpha(0^-)}{s^2 + 4s + 6}$. The zero-state response transform is $\frac{-10s - 12}{s^2 + 4s + 6}\Delta(s)$. The transfer function from $\delta_t$ to $\alpha_t$ is $\frac{-10s - 12}{s^2 + 4s + 6}$. This has one zero at $s = 1.2$ and poles at $s = -2 \pm j\sqrt{2}$.
Part ii: Using your answer from Part i or otherwise, show that the impulse response from $\delta_t$ to $q_t$ is given by

$$h_{q,t} = -5\sqrt{2}e^{-2t}\sin(\sqrt{2}t)1(t).$$

The impulse response from $\delta_t$ to $\alpha_t$ is given by [Do not show this.]

$$2\sqrt{33}e^{-2t}\cos(\sqrt{2}t - \text{atan2}(2\sqrt{2}, -5))1(t).$$

These impulse responses are shown in Figure 2 below. Interpret these impulse responses physically.

![Pitch rate and angle of attack impulse responses from elevator angle](image)

Figure 2: Impulse responses from elevator to pitch rate (solid) and angle of attack (dashed)

The impulse response is computed as the response with zero initial conditions and $\Delta(s) = 1$, i.e. the inverse Laplace transform of the transfer function. This has transform

$$-\frac{10}{j2\sqrt{2}} = \frac{C}{s + 2 - j\sqrt{2}} + \frac{C}{s + 2 + j\sqrt{2}}.$$

$$C = \lim_{s \to -2 + j\sqrt{2}} (s + 2 - j\sqrt{2}) \times -\frac{10}{(s + 2 - j\sqrt{2})(s + 2 + j\sqrt{2})} = -\frac{10}{j2\sqrt{2}}.$$

Thus the inverse transform is given by

$$h_{q,t} = \left[ -\frac{10}{j2\sqrt{2}}e^{-2t+j\sqrt{2}t} - \frac{10}{j2\sqrt{2}}e^{-2t-j\sqrt{2}t} \right] 1(t),$$

$$= \frac{10}{\sqrt{2}}e^{-2t} \frac{1}{2j} \left[ e^{j\sqrt{2}t} - e^{-j\sqrt{2}t} \right] 1(t),$$

$$= -5\sqrt{2}e^{-2t}\sin(\sqrt{2}t)1(t).$$

Physically, it is impossible to impose an impulse into the elevator – something would break. But we see from the mathematical model that doing so would cause an immediate step change in the angle of attack and a slower change in the pitch angle rate. The dependence of the angle of attack on the elevator rate is what causes (mathematically) the step change in angle of attack. This rapid time-scale is not very physical, since this is a linear model. On the longer time-scale we see that the system is stable and so both the pitch angle rate and angle of attack return to zero. That is, once the elevator is returned to zero, the plane resumes straight flight at a new pitch angle – only the pitch angle rate goes back to zero. If we started level before the impulse we are now flying straight at an angle downwards.

Part iii: Determine the long-term, i.e. $t \to \infty$, behavior of the pitch rate and of the angle of attack of the aircraft to a unit step in the elevator angle. Show your working for this and state what conditions need to be verified at each stage.
Interpret physically this long-term behavior of the aircraft to a step in the elevator angle. Do these computed responses make physical sense?

Since the system is stable (all poles in the open left half-plane), we can ignore the zero-input response in determining the behavior at time tending to infinity. Further, because of stability, we can use the Final Value Theorem to evaluate the response to a step input, since the response transform will have all poles in the left half-plane except for the single pole at \( s = 0 \) due to the step elevator input signal.

\[
\lim_{t \to \infty} q_t = \lim_{s \to 0} s Q(s) = \lim_{s \to 0} s \frac{-10}{s^2 + 4s + 6} \frac{1}{s} = -\frac{10}{6},
\]
\[
\lim_{t \to \infty} \alpha_t = \lim_{s \to 0} s A(s) = \lim_{s \to 0} s \frac{-10s - 12}{s^2 + 4s + 6} \frac{1}{s} = -2.
\]

As before with the impulse response, on the long time-scale this makes sense. The plane will tend to having a constant pitch rate. That is, the plane’s pitch angle will be decreasing linearly and the plane will execute a sequence of loops ... assuming its initial altitude was high enough. The angle of attack will also tend to a constant, which is to say that as it executed these loop maneuvers the plane’s nose will always be pointing below the tangential velocity vector, which is what happens in such aerobatics. It makes sense.
Question 2 — Cruise control dynamics and the effect of feedback

Part i: The dynamics of a car’s speed, \( v_t \), are described by the following ordinary differential equation

\[
M \dot{v}_t = -av_t + bu_t - d_t. \tag{3}
\]

Here: \( M \) is the car’s mass; the term \(-av_t\) is the drag; \( bu_t \) is the force due to the input fuel flow; and \( d_t \) is the influence of the terrain, regarded here as a disturbance. \( M, a, b \) are all positive.

Suppose that the fuel flow is chosen as a constant value \( u_t = \bar{u} \) and the disturbance is zero, \( d_t = 0 \). Show that the \( \star \) steady-state value of the speed is given by

\[
\lim_{t \to \infty} v_t = \bar{v} = \frac{b}{a} \bar{u}.
\]

What are the requirements on the values of the parameters \( a \) and \( b \) for this result to hold? \( \star \)

The system (3) has a pole at \( s = -a/M \). Since the parameters \( M, a, \) and \( b \) are all positive, the pole is in the open left half-plane and the system is stable. Therefore we can apply the Final Value Theorem to determine the steady-state value of the speed to a step input. Further, because of stability, we can ignore the zero-input response due to initial conditions, since it tends to zero as \( t \to \infty \). So, using Laplace transforms,

\[
\lim_{t \to \infty} v_t = \lim_{s \to 0} s \frac{b/M \bar{u}}{s + a/M} = \frac{b}{a} \bar{u}.
\]

A smarter way forward would be to use the stability to argue that at steady state \( \dot{v}_t = 0 \) so that \( 0 = -av_t + b\bar{u} \). This gives the same answer much more easily.

The central requirement on the parameters is that \( a/M \) should be positive, since we rely on stability for the existence of a steady-state solution. The value of \( b \) is immaterial to the analysis.

Part ii: With these definitions of \( \bar{u} \) and corresponding \( \bar{v} \), define new variables

\[
\tilde{v}_t = v_t - \bar{v},
\]

\[
\tilde{u}_t = u_t - \bar{u}.
\]

These new variables now describe the variation of fuel flow and speed from nominal constant values.

Show that \( \star \)

\[
M \dot{\tilde{v}}_t = -a\tilde{v}_t + b\tilde{u}_t - d_t. \tag{4}
\]

Differentiating, we see that \( \dot{\tilde{v}}_t = \tilde{v}_t \), since \( \bar{v} \) is a constant. So

\[
M \dot{\tilde{v}}_t = -av_t + bu_t - d_t,
\]

\[
= -av_t + b(u_t + \bar{u}) - d_t,
\]

\[
= -av_t + b\tilde{u}_t + a\bar{v} - d_t,
\]

\[
= -a(v_t - \bar{v}) + b\tilde{u}_t - d_t,
\]

\[
= -a\tilde{v}_t + b\tilde{u}_t - d_t.
\]

Show that the steady-state behavior of this speed variation, \( \tilde{v}_t \), to a step disturbance of size \( D \) in \( d_t \) with the fuel flow held constant at \( u_t = \bar{u} \) is given by

\[
\lim_{t \to \infty} \tilde{v}_t = -\frac{D}{a}.
\]

Interpret physically what this is saying about the car’s behavior.
Since the $v_t$ system is stable, so it the $\dot{v}_t$ system, as the two differ simply by a constant offset. Taking $u_t = \dot{u}$ causes $\dot{u}_t = 0$ and so we are looking solely at the response to the disturbance, $d_t$. Using the Final Value Theorem or the above approach of setting $\dot{v}_t = 0$ we arrive at

$$\lim_{t \to \infty} \dot{v}_t = \lim_{s \to 0} s \frac{-1/M}{s + a/M} = -\frac{D}{a}.$$  

In terms of the car’s behavior, this means that an uphill slope causes the car’s speed to drop below its nominal value $\bar{v}$ by $D/a$ units. This makes sense.

**Part iii:** Suppose that we now regulate the fuel flow using feedback of the speed deviation signal. That is,

$$u_t = \bar{u} + K_p(\bar{v} - v_t),$$

or,

$$\dot{u}_t = -K_p \bar{v}_t.$$  

Show that (4) now becomes

$$M\dot{\bar{v}}_t = -(a + bK_p)\bar{v}_t - d_t. \tag{5}$$

Substituting for $\bar{u}_t$ in (4) yields

$$M\dot{\bar{v}}_t = -a\bar{v}_t - bK_p\bar{v}_t - d_t = -(a + bK_p)\bar{v}_t - d_t.$$  

Determine the steady-state response of $\bar{v}_t$ to a step of size $D$ in disturbance $d_t$. What are the conditions required of $K_p$ for this result to hold? Show that there is no value of $K_p$ which will force $\bar{v}_t \to 0$ as $t \to \infty$ with the constant terrain disturbance.

If the system is stable, then we can use the Final Value Theorem (or the zero derivative approach) to determine the steady-state value. The system has a single pole at $s = -a - bK_p$. So the condition for a steady-state value to exist is that $a + bK_p < 0$ or $K_p > -a/b$, since both $a$ and $b$ are positive. If this condition is satisfied, then the steady-state response to a step of size $D$ in $d_t$ is $\bar{v}_t = -D(a + bK_p)$. Clearly, the size of this steady-state error decreases as $K_p \to \infty$ but this is not a feasible choice. So for any finite value of $K_p$, the steady-state error is non-zero. In physical terms, this means that we always have an offset speed to a change in terrain slope.

Write the transfer function from $d_t$ to $\bar{v}_t$ from (5) and write the poles and zeros as a function of $K_p$. What is the nature of the frequency response of this system: highpass, lowpass, bandpass? DC gain? and what does this tell us about the response of the system to general disturbances $d_t$?  

The transfer function from $d_t$ to $\bar{v}_t$ is

$$\frac{-1/M}{s + (a + bK_p)/M}.$$  

This has no zeros and a single pole at $s = -(a + bK_p)/M$. This transfer function is a lowpass system, since the frequency response would be $-1/(s + (a + bK_p)/M)$, which is a constant, $-1/(a + bK_p)$, for small values of $\omega$ but tends to zero magnitude for large values of $\omega$. It has a -3dB point at $\omega = -(a + bK_p)/M$. So we see that the bandwidth increases with feedback gain $K_p$.

**Part iv:** Because of this last answer, we seek to use a more sophisticated controller – so called *proportional-plus-integral* (or PI) controller

$$\dot{u}_t = -K_p \bar{v}_t - K_i \int_0^t \bar{v}(\tau) \, d\tau. \tag{6}$$

Take the Laplace transform of (6) and write the transform, $\tilde{U}(s)$, of $\dot{u}_t$ in terms of the transform, $\tilde{V}(s)$, of $\bar{v}_t$. Also take
the Laplace transform of (4) with zero initial conditions. Use these transformed equations to show that
\[
\tilde{V}(s) = \frac{-s/M}{s^2 + s(a + bK_p)/M + bK_i/M} \times D(s).
\] (7)

Taking the Laplace transform of (6) we have the following
\[
\tilde{U}(s) = -K_p \tilde{V}(s) - K_i \frac{1}{s} \tilde{V}(s).
\]
Likewise, taking Laplace transforms of (4) with zero initial conditions, we have
\[
Ms \tilde{V}(s) = -a \tilde{V}(s) + b \tilde{U}(s) - D(s).
\]
Now substituting for \( \tilde{U}(s) \), we see
\[
Ms \tilde{V}(s) = -a \tilde{V}(s) - bK_p \tilde{V}(s) - bK_i \frac{1}{s} \tilde{V}(s) - D(s),
\]
\[
s \tilde{V}(s) + \frac{1}{M}(a + bK_p) \tilde{V}(s) + \frac{1}{M}bK_i \frac{1}{s} \tilde{V}(s) = -\frac{1}{M} D(s),
\]
\[
\begin{bmatrix} s^2 + \frac{a + bK_p}{M}s + \frac{bK_i}{M} \end{bmatrix} \tilde{V}(s) = -\frac{s}{M} D(s),
\]
\[
\tilde{V}(s) = \frac{-s/M}{s^2 + s(a + bK_p)/M + bK_i/M} \times D(s)
\]

Determine now the steady state error in \( \tilde{v}_i \) due to a step of size \( D \) in the disturbance \( d_t \). What are the conditions on \( K_p \) and \( K_i \) to make this hold? [Hint: a second-degree polynomial has all its roots in the open left half-plane if all of its coefficients are positive.]

In order for a steady-state solution to exist we need the transfer function poles to be in the open left half-plane. According to the hint, this will be true if \( K_p > -a/b \) and \( K_i > 0 \). Once this is achieved, we can evaluate the steady-state response of \( \tilde{v}_i \) to a step of size \( D \) in \( d_t \) using the final value theorem.
\[
\lim_{t \to \infty} \tilde{v}_i = \lim_{s \to 0} s \tilde{V}(s) = \lim_{s \to 0} s \frac{-s/M}{s^2 + s(a + bK_p)/M + bK_i/M} \times D(s) = 0,
\]
That is, the steady-state error to this terrain disturbance is zero and the car’s steady-state speed is \( \tilde{v} \). There is, of course, a transient response as well. The stability conditions above ensure that this happens.

Examine the transfer function from \( d_t \) to \( \tilde{v}_i \) from (7) and write the poles and zeros as a function of \( K_p \) and \( K_i \). What is the nature of the frequency response of this system: highpass, lowpass, bandpass? DC gain? How does this depend on the values \( K_p \) and \( K_i \)? What does this tell us about the response of the system to general disturbances \( d_t \)?

The transfer function has a single zero at \( s = 0 \) (which accounts for the zero-steady-state response to a step) and has poles at the zeros of \( s^2 + s(a + bK_p)/M + bK_i/M \). That is, at
\[
s = -\frac{a + bK_p}{2M} \pm \sqrt{\left( \frac{a + bK_p}{2M} \right)^2 - \frac{bK_i}{M}}.
\]
Looking at the transfer function in (7) and considering the case where \( s = j\omega \) with \( \omega \) small we see that the numerator tends to zero while the denominator is a positive constant. So the transfer function tends to zero at low frequencies. Similarly, taking \( \omega \) becoming very large, we see that the transfer function also tends to zero at very high frequencies. But the transfer function is not zero. So it must be bandpass.

This implies for general disturbances that low frequency (i.e. slow) terrain changes will have no effect on the car’s speed. Likewise, high frequency terrain changes (like tops on the road in Mexico) will not affect the car’s speed either. But there could be a mid range of frequencies which affects the car’s speed. I have an example of this from Goulburn in New South Wales Australia, where my cruise control routinely makes my car exceed the speed limit because of two hills creating an almost sinusoidal disturbance.
Question 3 — Discrete data analysis

We have a batch chemical process and are considering the estimation of a reagent which is consumed during operation. In batch operation, the reaction vessel is charged with known quantities of materials and proceeds to completion with the possible addition of reagents. The classical example of batch operation is the brewing of beer or the manufacture of wine, where the fermentation vessel is closed and fermentation takes place over a time horizon of roughly one week. Batch reactions are necessarily finite time.

Our process operation is dependent on the maintenance of a fixed concentration of one reaction component throughout the batch cycle. This component is, however, consumed slowly by the reaction. Accordingly, it must be added during the cycle. The measurement of the reagent concentration is carried out using a sensor which, because of degradation over time due to erosion of material (a metal), yields a measurement value (a threshold voltage) which drifts upwards roughly linearly over time.

The following two figures show the 999 points of time domain data in the first figure followed by the first 94 elements of the Discrete Fourier Transform of these 999 data plotted using the stem function. The rest of the bins are close to zero. The data are sampled every 40 seconds.

![Time series of reagent concentration in batch reaction](image1)

**Figure 3**: Batch process 999 data points in time domain.

**Part i**: Rewrite the scale of the time-domain plot in hours. Rewrite the scale of the zoomed DFT plot in cycles per hour. Particularly identify precisely the frequencies associated with bins 12, 22, and 40.

There are 999 samples taken every 40 seconds, which is one sample every two-thirds of a minute or 90 samples per hour. That is, the time scale stretches from 0 minutes to 998/90 ≈ 11.1 hours. The time axis is scaled by starting at zero and dividing by 90.

Likewise, the first bin of the DFT is zero frequency and the second bin is the fundamental frequency of 1 cycle in 11.1 hours. The twelfth bin will then represent 11 cycles in 11.1 hours or very close to once per hour. The 22nd bin will be 21 cycles in 11.1 hours or close to every 30 minutes. The 40th bin is 39 cycles in 11.1 hours.

**Part ii**: Identify and describe features visible in each of the plots. Consider whether these items are discernible in only
one plot or in both plots. [Be quantitative and as precise as possible. Marks will be awarded for quality of exposition and content. Imagine you are describing the signal and the DFT to your boss in an email, for example.]

Features in the time-domain plot:

1. non-zero average value roughly 0.5,
2. seeming slope upwards from beginning to end explained in the question as due to the consumption/erosion of an electrode in the sensor,
3. a considerable variation around this upwards slope and offset,
4. perhaps there is a an almost discernible cycling of about 21 times during the data,
5. there are some ramp-like increases starting around samples 150, 400, 525, 750

Features in the DFT plot

1. The first bin has a large value. This corresponds to the non-zero average value and to the first item observed in the time-domain plot. If we had its value, it would be 999 times the average offset.
2. The next three or four bins have a large but decreasing amplitude. This might correspond to the sawtooth wave nature of the time-domain data, if it were to be repeated. In this sense, this corresponds to the second item in the list above.
3. We see bins 12, 22 and 40 appearing to stand out from the others. The corresponding frequencies are identified as hourly, every thirty minutes and roughly every 17 minutes. The bin 22 data seems to be visible in the time-domain plot. The others cannot be discerned in the time-domain plot.

Part iii: Consider the 4-term sequence \( \{ x_t : t = 0, 1, 2, 3 \} = \{ 1, 1, -1, -1 \} \). Using the DFT formula,
Now note that show that the corresponding 4-term DFT is \( \{ c_k : k = 0, 1, 2, 3 \} = \{ 0, 2 \pm 2j, 0, 2 \pm 2j \} \).

Start of computing the sequences \( \{ e^{-j2\pi kt/4} : t = 0, 1, 2, 3 \} \)

\[
[k = 0] \quad \{ e^{-j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, 1, 1, 1 \},
\]

\[
[k = 1] \quad \{ e^{-j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, -j, -1, j \},
\]

\[
[k = 2] \quad \{ e^{-j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, -1, 1, -1 \},
\]

\[
[k = 3] \quad \{ e^{-j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, j, -1, -j \}.
\]

Then, taking the sequence \( \{ x_t : t = 0, 1, 2, 3 \} = \{ 1, 1, -1, -1 \} \) and using the DFT sum we have

\[
c_0 = x_0 + x_1 + x_2 + x_3 = 0,
\]

\[
c_1 = x_0 - jx_1 - x_2 + jx_3 = 2 - 2j,
\]

\[
c_2 = x_0 - x_1 + x_2 - x_3 = 0,
\]

\[
c_1 = x_0 + jx_1 - x_2 - jx_3 = 2 + 2j.
\]

Interpret each of these four values of \( c_k \) in terms of the properties of the \( x_t \) sequence.

Looking at the inverse DFT formula below, we see that this says that the sequence \( \{ x_t \} \), if it were to be repeated over and over again, would yield a periodic discrete signal with period 4 samples and with zero constant average value and no element at the Nyquist frequency. That is the sequence would be composed entirely of the fundamental frequency. This is indicated by the \( k = 1 \) and \( k = 4 - 1 = 3 \) values being non-zero and the others being zero.

To see this in more detail, note that \( \frac{1}{4}(2 - 2j) = \frac{\sqrt{2}}{2} e^{-j\pi/4} \). So

\[
x_t = \frac{1}{4} \left[ (2 - 2j)e^{j\frac{2\pi}{4}t} + (2 + 2j)e^{-j\frac{2\pi}{4}t} \right]
\]

\[
= \sqrt{2} \left[ e^{j\frac{\pi}{4}(t - \frac{1}{4})} + e^{-j\frac{\pi}{4}(t - \frac{1}{4})} \right],
\]

\[
= \sqrt{2} \cos \left( \frac{\pi}{4} t - \frac{\pi}{4} \right).
\]

Now note that

\[
x_0 = \sqrt{2} \cos \left( \frac{\pi}{2} 0 - \frac{\pi}{4} \right) = \sqrt{2} \cos \left( -\frac{\pi}{4} \right) = 1,
\]

\[
x_1 = \sqrt{2} \cos \left( \frac{\pi}{2} 1 - \frac{\pi}{4} \right) = \sqrt{2} \cos \left( \frac{\pi}{4} \right) = 1,
\]

\[
x_2 = \sqrt{2} \cos \left( \frac{\pi}{2} 2 - \frac{\pi}{4} \right) = \sqrt{2} \cos \left( \frac{3\pi}{4} \right) = -1,
\]

\[
x_3 = \sqrt{2} \cos \left( \frac{\pi}{2} 3 - \frac{\pi}{4} \right) = \sqrt{2} \cos \left( \frac{5\pi}{4} \right) = -1,
\]

So the signal is composed of the fundamental only. No constant dc term and no term like \( (-1)^t = e^{j\pi t} \) at the Nyquist frequency.

If you have time, show that \( \{ x_t \} \) is yielded by the inverse DFT formula

\[
x_t = \frac{1}{4} \sum_{k=0}^{3} c_k \exp \left( j \frac{2\pi t}{4} k \right),
\]
Repeating the above calculation for \( \{ e^{j2\pi kt/4} : t = 0, 1, 2, 3 \} \)

\[
\begin{align*}
[k = 0] & \quad \{ e^{j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, 1, 1 \}, \\
[k = 1] & \quad \{ e^{j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, j, -1, -j \}, \\
[k = 2] & \quad \{ e^{j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, -1, 1, -1 \}, \\
[k = 3] & \quad \{ e^{j2\pi kt/4} : t = 0, 1, 2, 3 \} = \{ 1, -j, -1, j \}.
\end{align*}
\]

We have

\[
\begin{align*}
x_0 &= \frac{1}{4} [0 + (2 - 2j) + 0 + (2 + 2j)] = 1, \\
x_1 &= \frac{1}{4} [0 + (2 - 2j)j + 0 - (2 + 2j)j] = 1, \\
x_2 &= \frac{1}{4} [0 - (2 - 2j) + 0 - (2 + 2j)] = -1, \\
x_3 &= \frac{1}{4} [0 - (2 - 2j)j + 0 + (2 + 2j)j] = -1,
\end{align*}
\]