

A Constrained Model-Predictive Approach to Coordinated Control

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Abstract

We consider constrained Model Predictive Control (MPC) as a local control law for application in coordinated control of a group of distributed autonomous systems. Interactions between the systems are captured via constraints. Restricted communication bandwidth is available and forces the subsystems to avail themselves of limited-quality estimates of their neighbors' states in computing this constrained control. Stability theorems are derived for the subsystems and illustrate the conditions required of the constraint set in order to guarantee local stability or convergence to a target state. If these conditions are met for all subsystems, then this stability is inherited by the overall system. Disturbance attenuation, or "string stability", is studied in this framework and it is shown that inactivity of the MPC constraints implies stability. This then provides a connection between control objective, communications resource assignment and performance. A one-dimensional vehicle example is computed to crystallize ideas.

1 Introduction

Constrained Model Predictive Control (MPC) is advanced as an effective approach to the design of coordinated control and communications for multiple autonomous systems. The central ideas are the use of local MPC at each distributed subsystem with the interaction managed via "no-collision" constraints between the subsystems being included into the MPC. Limited-bandwidth communication between the subsystems is available and its resource assignment is coupled to the achievable global system performance.

In considering distributed coordination, it is natural to identify tasks which are global, i.e. shared among all subsystems, in nature and those which are local or limited in their domain. Here we specify the global tasks to include those design elements which might sensibly be pre-computed and are known by all subsystems; reference trajectories, constraint specification, control law, observer structure and design, communications resource assignment, maps and other functional descriptions of the environment. By contrast, local tasks would be those associated with real-time and on-line calculation and the interaction between subsystems; the MPC control solution computation, observer calculation of state estimates, operation of communications links.

Our approach in this paper is to explore the application of local constrained MPC control laws on the subsystems and to establish conditions on the constraints to permit guaranteed convergence to a local target trajectory or point. Thereby, if the global set of constraints is such as to permit this local satisfaction, then system stability and convergence follow. Properties of the constraint set flow from the specification of reference trajectories and of the communications bandwidth, as will be shown. This establishes a link between control objective and communication. To help fix ideas we focus on a vehicle coordination problem and demonstrate concepts using simulations of a one-dimensional example.

Design via constraints is an attractive approach to control and coordination. In particular, the recent interest in the application of MPC (Maciejowski, 2002; Goodwin et al., 2004) reflects the appeal of constraints as a design method

for controlling systems. In spite of the approach to the solution of the constrained control via optimization with penalty or barrier functions, the attraction of MPC rests in the separation of constraints and the objective function. In vehicle control problems, recent work of (Gerdes and Rossetter, 2001) indicates that common coordination tasks such as lane-keeping and distance-keeping are well captured by potential functions which may be interpreted as the action of constraints on the controlled driving dynamics.

The thesis of this paper is that coordinated control has a natural specification via imposed constraints between neighbors' actions. An MPC-like constrained control is a natural approach to developing local feedback control laws using communicated state information from the neighboring systems. We separate the global pre-computed tasks to include the determination of target (or reference) trajectories for each subsystem and use this to inform the constraint management and communication resource assignment. The sensing on board each subsystem will be only self-sensing, say position, and some level of communication of this information is permissible to designated others. This helps crystallize the importance of the communication resource assignment task in coordinated control.

In this paper, we first discuss the formulation of a typical vehicle formation problem in Section 2. In Section 3, the convergence of the entire formation is deduced from the local stability guaranteed by the local constrained MPC for deterministic, time-invariant/varying case. For the case with stochastic disturbances, such as communication channel noises, the no-collision constraint of MPC is modified to accommodate the uncertainty introduced by the disturbances. String stability, which describes the propagation of the disturbances in the formation, is discussed. String stability can be guaranteed via decoupling the vehicles' closed-loop dynamics, i.e. using fixed reference trajectories in MPC cost functions and keeping constraints inactive. In Section 4, the working problem is a simple 1-Dimensional vehicle formation control problem. In each local MPC controller, the vehicles are decoupled in their local criteria while having the modified no-collision constraints coupling the neighbours. A leader-follower strategy with the vehicles coupled in the criterion functions is included for the purpose of comparison. Two performance indices are evaluated; string stability and inter-vehicle spacing. The relation between the communication channel noises and the control performance is clear from the simulations. This relation is also a guide to the design of the reference trajectories and the assignment of the communication bandwidth. The conclusions follow in Section 5. The detailed formulation of the simulation is given in the Appendix.

2 Multi-vehicle Coordination Problem

Vehicle formation control has become an active topic in the control community with many possible applications, for example, robots (Balch and Arkin, 1998; Desai et al., 2001), unmanned aerial vehicles (Stipanović et al., 2004) or spacecraft (Beard et al., 2001), and automated highway systems (Varaiya, 1993; Swaroop and Hedrick, 1999), just to mention a few. In this paper, we consider a vehicle fleet consisting of p vehicles with concatenated state $X_k = \left(x_k^1, x_k^2, \dots, x_k^p\right)^T$. The dynamics of the i th vehicle are given by

$$x_{k+1}^i = f^i(x_k^i, u_k^i) + w_k^i, \tag{1}$$

$$z_k^i = \pi^i(X_k) + v_k^i. \tag{2}$$

- x_k^i is the state vector, u_k^i is the local control, and w_k^i is the process disturbance impinging on the state of Vehicle i . Here the $f^i(\cdot)$ s need not all be the same.
- The measurement function $\pi^i(\cdot)$ captures the communication link selections to Vehicle i and the nature of the communicated information, which could include the states of a subset of vehicles, the control values of a subset of vehicles, or the predicted future control and state values as in (Dunbar and Murray, 2004).
- The noise term v_k^i models both the measurement noise and the inaccuracy of the data communicated via π^i . This latter part of v_k^i is due to finite bit-rates being assigned to the communication links carrying the data and is practically very well modelled by uniformly distributed white noise. The effects of channel noise could similarly be included.

We define the term *information architecture* to mean the complete set of assignments of bit-rates (including zero-bit rate links to represent un-communicated data) to communication links for communicated data between vehicles and the selection of which data are communicated. Typically, the information architecture is constrained by channel bandwidth.

A typical vehicle formation task is described as: given a target formation $X^* = (x^{1*T}, x^{2*T}, \dots, x^{p*T})^T$, the local controllers should steer the entire formation so that $X_k \rightarrow X^*$ while avoiding collision. Some other performance requirements may also be added, such as the speed of convergence or energy usage.

For a large formation, a decentralized strategy is preferred so that the online computation speeds match the dynamics of the vehicles and we see this as dividing into two design phases:

- (1) The first part is the design of the information architecture, which determines the off-line shared information (such as a virtual leader (Leonard and Fiorelli, 2001)) and the necessary localized information flows for each vehicle to guarantee the fulfilment of the global task. It has been shown that Graph Theory tools lend themselves naturally in the information flow design problem (Muhammad and Egerstedt, 2004). The stability of the formation can be directly related the algebraic properties of the information flow graphs as shown in (Fax and Murray, 2002, 2004; Moreau, 2005). Perfect measurement is assumed in the works above. When information is obtained by sensors or communication channels, there will be noises associated with the information. This raises another design issue in the information architecture design — the design of the accuracy of the local information, which is directly related to the cost of the communication devices or sensors.

In this paper, we assume the architecture of the information flows is given. The net quantified effect of the specification of the information architecture is captured through the computed uncertainty of the global state, \hat{X}_{k+j} , evaluated at time k and at Vehicle i using the sensed and communicated data. Denote this $\hat{X}_{k+j|k}^i$. The uncertainty may be modelled in a number of ways, notably;

- the support of the density of $\hat{X}_{k+j|k}^i$, or
- the covariance of $\hat{X}_{k+j|k}^i$.

The global state estimate and its information-architecture-dependent uncertainty will pass into the formulation of the collision avoidance MPC controller. In Section 4, we will discuss the relation between the information inaccuracy and the control performance in a simple one-dimensional vehicle control problem. This provides a guide to the design.

- (2) The second part is the design of the local controllers, which should lead the vehicles to their targets while avoiding collisions. MPC is an appealing approach because of its capability of constraint-handling, familiarity with its real-time implementation and the solid theoretical results on stability and feasibility (Mayne et al., 2000). There are previous works on the application of distributed MPC to vehicle formation control. In (Keviczky et al., 2004), a decentralized MPC formulation is posed with the local cost functions derived from a global MPC problem and the vehicles are coupled in both cost functions and constraints. Feasibility becomes difficult and the stability is not proven. In (Dunbar and Murray, 2004), the vehicles exchange their optimizing future trajectories at each sampling time. The stability of the formation can be guaranteed by including explicitly a compatibility constraint in the local MPC controllers. These works assume perfect state information.

In this paper, we have the vehicles decoupled in their local cost functions by introducing to each vehicle a pre-computed reference trajectory to follow. The vehicles are coupled via the no-collision constraints in their local MPC. These constraints incorporate the state estimates, $\hat{X}_{k+j|k}^i$, and the uncertainty characterization flowing from the information architecture. By applying the strategy in (Yan and Bitmead, 2005), the local constraints may be reposed to accommodate the quantified uncertainty. Furthermore, these modified constraints relate the information quality and the control performance as will be shown in Section 4.

3 Local Model Predictive Control Law

To accomplish the coordination task stated in Section 2, each vehicle should approach its own target state x^{i*} while complying with constraints. We propose to use constrained MPC as the local control law for this constrained

problem. A typical local constrained MPC problem for the i th vehicle is:

$$\begin{aligned} \min_{u_{k,k}^i, \dots, u_{k+N-1,k}^i} J_k^i(x_k^i, u_{k,k}^i, \dots, u_{k+N-1,k}^i) &= \min F_k^i(x_{k+N}^i) + \sum_{j=0}^{N-1} l_k^i(x_{k+j}^i, u_{k+j}^i), \\ \text{subject to: } x_{k+j+1}^i &= f^i(x_{k+j}^i, u_{k+j}^i) + w_{k+j}^i, \\ u_{k+j}^i &\in U_{k+j}^i, \\ x_{k+j}^i &\in X^i(k+j), \\ x_{k+N}^i &\in X_f^i(k), \end{aligned} \quad (3)$$

where the criterion function consists of the sum of step costs $l_k^i(\cdot, \cdot)$ and a terminal state cost $F_k^i(\cdot)$. The feasible set of controls is the compact, convex set U_k^i and the feasible set of states is the closed, convex set $X^i(k)$ which should be determined online by the available information π_k^i . The terminal state constraint set $X_f^i(k)$ is the key to guaranteeing the stability of the MPC problem. The time index k in these constraint sets indicates that they can be time dependent. (The initial state x_k^i should be given by some estimator/observer based on z_k^i .)

3.1 Formulation and Stability Analysis of the Deterministic Case

In the deterministic case first, the process noises w_k^i and the measurement noises v_k^i in (3) are zero. A thorough stability discussion of existing constrained MPC approaches can be found in (Mayne et al., 2000). The MPC problem in this paper is slightly different because the target state x^{i*} may be outside the constraint sets X^i and X_f^i . This may occur in the vehicle coordination problem.

Consider the time-invariant case first, meaning that $X^i(k)$ and $X_f^i(k)$ become X^i and X_f^i in the local MPC problem (3).

Definition 1 Functions $\bar{l} : R^n \times R^m \rightarrow R$ and $\bar{F} : R^n \rightarrow R$, where n and m are the dimensions of x_k^i and u_k^i ; $\bar{l} > 0$, $\bar{F} > 0$ except $\bar{l}(x^{i*}, u^{i*}) = \bar{F}(x^{i*}) = 0$; $\bar{l}(x^i, u^i) \uparrow \infty$ as $\|(x^i, u^i) - (x^{i*}, u^{i*})\| \uparrow \infty$ and $\bar{F}(x^i) \uparrow \infty$ as $\|x^i - x^{i*}\| \uparrow \infty$.

Using the stationary running state and control constraint sets X^i and U^i , define the terminal state constraint set X_f^i to be the smallest positively-invariant set of the controlled states contained in the set of the \bar{F}^i -closest feasible points to the target state. Denote by U_f^i the set of controls which would maintain x_k^i in X_f^i .

Thus, if the target were feasibly reachable and there exists a control u_f^i satisfying $x^{i*} = f^i(x^{i*}, u_f^i)$, then $X_f^i = \{x^{i*}\}$ and $U_f^i = \{u^{i*}\}$. If x^{i*} were feasibly reachable but no holding control as above existed, then X_f^i would be a neighbourhood around x^{i*} and U_f^i the set of controls which kept X_f^i positively-invariant. If x^{i*} were not feasibly reachable, then X_f^i would be a positively-invariant neighbourhood of feasible points as close as possible to x^{i*} and U_f^i the set of controls which hold x_k^i in X_f^i .

Theorem 1 If the MPC problem (3) satisfies the following conditions:

- (1) U^i is compact and convex, X^i is closed and convex;
- (2) the step cost and terminal cost functions are $l^i(x_k^i, u_k^i) = \bar{l}^i 1_{\{x_k^i \notin X_f^i, u_k^i \notin U_f^i\}}$, $F^i(x_k^i, u_k^i) = \bar{F}^i 1_{\{x_k^i \notin X_f^i\}}$; where $1_{\{\cdot\}}$ is the indicator function;
- (3) $f^i(\cdot)$, $\bar{l}^i(\cdot, \cdot)$, and $\bar{F}^i(\cdot)$ are continuous;

then, provided an initial feasible solution exists, the MPC controller yields a feasible control law $\{u_k^i(x_k^i)\}$ and trajectory $\{x_k^i\}$ with $\limsup_{k \rightarrow \infty} \{x_k^i\} \subset X_f^i$.

Proof: To show that the closed-loop system converges to the terminal constraint set X_f^i , we first to show that the criterion function is a strictly decreasing function when the initial state is in the region $X^i \setminus X_f^i$.

Suppose the state at time k is any $x_k^i \in X^i \setminus X_f^i$, the MPC problem (3) yields an optimizing control sequence $\mathbf{u}_k^{\mathbf{i},\text{opt}} = \{u_{k,k}^{i,\text{opt}}, u_{k+1,k}^{i,\text{opt}}, \dots, u_{k+N-1,k}^{i,\text{opt}}\}$, the corresponding state trajectory is $\mathbf{x}_k^{\mathbf{i},\text{opt}} = \{x_{k+1,k}^{i,\text{opt}}, x_{k+2,k}^{i,\text{opt}}, \dots, x_{k+N,k}^{i,\text{opt}}\}$, and the associated cost is

$$J^{i,\text{opt}}(x_k^i) = J^i(x_k^i, \mathbf{u}_k^{\mathbf{i},\text{opt}}) = \sum_{j=0}^{N-1} l^i(x_{k+j,k}^{i,\text{opt}}, u_{k+j,k}^{i,\text{opt}}) + F^i(x_{k+N,k}^{i,\text{opt}}).$$

Since the solution $(\mathbf{u}_k^{\mathbf{i},\text{opt}}, \mathbf{x}_k^{\mathbf{i},\text{opt}})$ is feasible, it follows that $x_{k+N,k}^{i,\text{opt}} \in X_f^i$ and by the definition of X_f^i , there is a $u_{f,k+N}^i$ such that $x_{k+N+1,k}^i = f^i(x_{k+N,k}^{i,\text{opt}}, u_{f,k+N}^i) \in X_f^i$. Define

$$\tilde{\mathbf{u}}_{k+1}^{\mathbf{i}} = \{u_{k+1,k}^{i,\text{opt}}, u_{k+2,k}^{i,\text{opt}}, \dots, u_{k+N-1,k}^{i,\text{opt}}, u_{f,k+N}^i\}, \tilde{\mathbf{x}}_{k+1}^{\mathbf{i}} = \{x_{k+2,k}^{i,\text{opt}}, x_{k+3,k}^{i,\text{opt}}, \dots, x_{k+N,k}^{i,\text{opt}}, x_{k+N+1,k}^i\},$$

and their associated cost

$$\tilde{J}^i(x_{k+1,k}^{i,\text{opt}}, \tilde{\mathbf{u}}_{k+1}^{\mathbf{i}}) = \sum_{j=1}^{N-1} l^i(x_{k+j,k}^{i,\text{opt}}, u_{k+j,k}^{i,\text{opt}}) + l^i(x_{k+N,k}^{i,\text{opt}}, u_{f,k+N}^i) + F^i(x_{k+N+1,k}^i).$$

Note that, $l^i(x_{k+N,k}^{i,\text{opt}}, u_{f,k+N}^i) = F^i(x_{k+N+1,k}^i) = F^i(x_{k+N,k}^{i,\text{opt}}) = 0$, it follows that

$$J^{i,\text{opt}}(x_k^i) - \tilde{J}^i(x_{k+1,k}^{i,\text{opt}}) = l^i(x_k^i, u_k^{i,\text{opt}}) > 0.$$

The MPC strategy takes $u_k^i = u_k^{i,\text{opt}}$, thus, $x_{k+1}^i = x_{k+1,k}^{i,\text{opt}}$. It follows that $(\tilde{\mathbf{u}}_{k+1}^{\mathbf{i}}, \tilde{\mathbf{x}}_{k+1}^{\mathbf{i}})$ is a feasible solution for the MPC problem for the next step and by optimality, the optimized cost $J^{i,\text{opt}}(x_{k+1}^i) \leq \tilde{J}^i(x_{k+1,k}^{i,\text{opt}}) < J^{i,\text{opt}}(x_k^i)$. Therefore,

$$J^{i,\text{opt}}(x_{k+1}^i) - J^{i,\text{opt}}(x_k^i) \leq \tilde{J}^i(x_{k+1,k}^{i,\text{opt}}) - J^{i,\text{opt}}(x_k^i) < -l^i(x_k^i, u_k^{i,\text{opt}}).$$

By induction, we have

$$0 \leq J^{i,\text{opt}}(x_{k+1}^i) < J^{i,\text{opt}}(x_0^i) - \sum_{j=0}^k l^i(x_j^i, u_j^{i,\text{opt}})$$

hold for arbitrarily large k . This requires that the series $\sum_{j=0}^k l^i(x_j^i, u_j^{i,\text{opt}})$ converges (to some value no larger than $J^{i,\text{opt}}(x_0^i)$), therefore, $l^i(x_j^i, u_j^{i,\text{opt}})$ converges to 0. By the definition of $l^i(\cdot, \cdot)$, (x_k^i, u_k^i) converges to the set $X_f^i \times U_f^i$. ■

The asymptotic stability implies that x_k^i converges to X_f^i as $k \rightarrow \infty$. This means that if the target x^{i*} is inside the constraint set X^i , then x_k^i will converge to a neighbourhood of x^{i*} . If x^{i*} is outside the constraint set, then x_k^i will converge to the set X_f^i , the closest feasible points to the target. In the vehicle formation problem, this implies the following corollary.

Corollary 1 *For a p -vehicle formation with each vehicle running an MPC controller that satisfies the conditions of Theorem 1, each vehicle in the formation will either converge to its target x^{i*} , or to the set of the closest points to the target defined by X_f^i . In particular, if the terminal state constraint set is $X_f^i = x^{i*}$, then the local asymptotic stability implies the global fleet formation stability.*

For the time-varying case, the time-dependence of the criterion function and the constraint sets may invalidate the proof from the time-invariant case. Since the tail of the last optimizing sequence might not be feasible for the new problem and, even if feasible, the new criterion function might not be decreasing.

If the cost function in the MPC problem is fixed, then one may still have formation stability by managing the time-varying constraint sets $X^i(k)$ and $X_f^i(k)$ as in the following theorem.

Theorem 2 *If there exists a finite time T so that $\forall k \geq T$:*

- (1) $x^{i*} \in X^i(k+j)$, $X_f^i(k)$ is a positively invariant neighbourhood of x^{i*} ; [respectively $X_f^i(k) = \{x^{i*}\}$, if this singleton is positively invariant];
- (2) the cost function stays the same after time T , i.e. $l_k^i(\cdot, \cdot) = l_T^i(\cdot, \cdot)$ and $F_k^i(\cdot) = F_T^i(\cdot)$;
- (3) $X^i(k) \subseteq X^i(k+1)$;

then, the MPC controller can stabilize x_k^i to a neighbourhood of the target x^{i*} [respectively to the target x^{i*}] asymptotically.

The proof is similar to the proof of Theorem 1. The Condition 3 guarantees the feasibility of the previous optimizing trajectory for the next step of optimization. The formation stability can be achieved via two-stage manipulation. First, the vehicles should move to have the targets exposed to them by a certain time T . After that time, if the local MPC of each vehicle satisfies Theorem 2, then the entire formation will converge to the target formation.

Condition 3 in Theorem 2 requires non-shrinking state constraint set so that the feasibility and stability can be guaranteed. In case that, at some time k , one vehicle were assigned with an overly large $X^i(k)$, Condition 3 would require all other vehicles to leave this large set $X^i(k)$ intact. This may lead to limited behavior of the rest of the formation or even may cause infeasibility at other vehicles. Hence, this condition should be guaranteed by careful off-line design of the reference trajectories, which might itself be a difficult problem. The following condition may be used to replace the Condition 3:

(3') define $C^i(k)$ to be the convex hull of the set $\{x_{k+2,k}^{i,opt}, x_{k+3,k}^{i,opt}, \dots, x_{k+N,k}^{i,opt}\}$ from the optimization at time k , the state constraint set at time $k+1$ should satisfy $C^i(k) \subseteq X^i(k+1)$.

Comparing to the Condition 3, Condition 3' allows the real-time change (shrinking) of the state constraint set $X^i(k)$ and the convergence of vehicle i to the target x^{i*} is guaranteed as long as the set $C^i(k)$ is preserved from time k to time $k+1$, which can be less demanding than Condition 3. Condition 3' should be guaranteed online by posing the reservation of $C^i(k)$ as constraints to the neighbouring vehicles, but this will raise the load of the real-time communication and the computation.

3.2 Formulation of the Stochastic Case

Disturbances to the measurement of the neighboring vehicles' states may exist and usually are caused by the communication finite-bit quantization effect and channel noises. Random variables with compact support can be a good model for this type of disturbances. In this case, instead of the exact π_k^i , the available measurement at the i th vehicle is

$$z_k^i = \pi_k^i + v_k^i.$$

Thus, state estimators for neighbours' states based on the collection of z_k^i are necessary. If the process noise w_k^i is also in the picture, each vehicle needs a self-estimator as well. The local MPC problem must be posed with these estimates.

If the estimates were used to replace the true values in the constrained MPC problem, then the optimizing control sequence could guarantee that the constraints were satisfied only by the estimated state process. If such a control law were applied to the vehicle formation, then the disturbance could drive the real state to violate the constraints even though the estimated state satisfied the constraints.

In order to use the estimates in the MPC problem, the constraints for the deterministic case should be tightened to reflect the necessary stand-off to buffer the uncertainty. We propose the same strategy as in (Yan and Bitmead, 2003) which applied the MPC formulation posed in (Yan and Bitmead, 2005) to the vehicle formation control problem.

The MPC problem is formulated in a stochastic way:

$$\min_{u_k^i, \dots, u_{k+N-1}^i} J^i(x_k^i, u_k^i, \dots, u_{k+N-1}^i) = \min E_{\mathcal{Z}_k^i} \left\{ \sum_{j=0}^{N-1} l_k^i(x_{k+j}^i, u_{k+j}^i) + F_k^i(x_{k+N}^i) \right\},$$

subject to: $x_{k+j+1}^i = f^i(x_{k+j}^i, u_{k+j}^i) + w_{k+j}^i,$

$u_{k+j}^i \in U^i,$

$P_{\mathcal{Z}_k^i}(x_{k+j}^i \in X^i(\pi_k^i)) = 1,$ (4)

$P_{\mathcal{Z}_k^i}(x_{k+N}^i \in X_f^i) = 1.$ (5)

The set $\mathcal{Z}_k^i = \{z_0^i, z_1^i, \dots, z_k^i, u_0^i, u_1^i, \dots, u_k^i\}$ is the collection of measurements and controls collected by Vehicle i up to time k . The *almost sure* state constraints (4) and (5) are posed in conditional probability given \mathcal{Z}_k^i , since the disturbances are modelled as random variables with compact support.

In (Yan and Bitmead, 2005), it has been shown that, for a linear system with linear inequality constraints and normally distributed noise signals, the probabilistic state constraints can be recast as deterministic constraints on the predicted states. Hence, the probabilistically constrained MPC can be recast as a deterministic quadratic programming problem for which efficient algorithms exist. The same procedure is applicable with noises of compact support and almost sure constraints. The inequality constraint values are recast from probabilistic constraints on x_{k+j}^i to deterministic constraints on $\hat{x}_{k+j|k}^i$, the state predictions. These modified constraints are tighter than the originals and involve the state estimate's support or covariance. This provides a direct connection between information architecture and control performance. The example of Section 4 will illustrate this.

3.3 String Stability

Theorems 1 and 2 deal with the asymptotic stability of the constrained MPC problem in the absence of disturbance on the states or measurement. (Swaroop and Hedrick, 1996) defined string stability of a platoon of vehicles to be the re-convergence of the entire platoon to their normal operating positions after an isolated disturbance to each of the vehicles. Similarly, (Yanakiev and Kanellakopoulos, 1996) require diminution of such disturbances along the one-dimensional vehicle string from the lead.

In our formulation, disturbances enter through state process noise and through communications quantization and channel noises, which have the prospect of introducing disturbances throughout the fleet. Thus the propagation of these disturbances needs consideration.

Theorem 3 *If the disturbances w_k^i, v_k^i do not cause the MPC no-collision constraints to become active then the fleet state X_k converges to the neighbourhood of X^* given by the decoupled controlled response of the individual vehicles to their self-noises.*

Proof: If the no-collision MPC constraints are inactive, then $u_k^i = u_k^i(\hat{x}_{k|k}^i) = u_k^i(\hat{x}_{k|k}^i(w_k^i, v_k^i))$. Thus, x_k^i depends only on self-noises and follows a decoupled response. ■

This theorem gives some insight into stability of the fleet. Provided disturbances are not so large as to cause the strong dependence on (uncertain) positions of neighbours, there is no effect. Likewise if the stationary positions X^* are feasibly achievable with inactive state constraints in the limit, then there will be a time beyond which sufficiently small disturbances will not propagate. There are clear ties to the concept of weak coupling systems and the string instability of (Swaroop and Hedrick, 1996). Here the key observation is that the condition of Theorem 3 for inactivity of no-collision constraints ties together fleet performance embodied in X^* and the information architecture captured through the tightened constraints, which reflect the support or covariance of the state estimate distribution. These inter-relations will become more evident in the next section.

4 Application to a 1-D Vehicle Formation Problem

4.1 1-D Vehicle Formation Problem

In this section, we consider a simple vehicle formation problem that consists of three locomotives on a single track operating with communications links seeking to maintain a one-dimensional formation.

Dynamics: The locomotives' dynamics are identical and are described by a simple integrator without process noise.

$$x_{k+1}^i = x_k^i + u_k^i, \quad i = 1, 2, 3,$$

where x_k^i is the absolute position of Vehicle i , and u_k^i represents its controlled velocity.

Information Architecture: Each vehicle has perfect knowledge of its own state and control. This information is communicated to the immediately following vehicle only so that,

$$z_k^i = \pi_k^i + \begin{pmatrix} \nu_k^i \\ \mu_k^i \end{pmatrix} = \begin{pmatrix} x_k^{i-1} \\ u_k^{i-1} \end{pmatrix} + \begin{pmatrix} \nu_k^i \\ \mu_k^i \end{pmatrix}, \quad i = 2, 3,$$

where ν_k^i and μ_k^i are white noises with $\nu_k^i \sim U[-\frac{V}{2}, \frac{V}{2}]$ and $\mu_k^i \sim U[-\frac{U}{2}, \frac{U}{2}]$, $U, V > 0$, capturing the communication noises and the communication bandwidth assignment.

Local MPC Problems:

$$\text{MPC-1:} \quad u_k^1 = \arg \min_{u_k^1} (x_{k+1}^1 - r_{k+1}^1)^2 \quad (6)$$

$$\text{MPC-2\&3:} \quad u_k^i = \arg \min_{u_k^i} E_{\mathcal{Z}_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \quad (7)$$

$$\text{[no-collision]} \quad \text{subject to: } P_{\mathcal{Z}_k^i} (x_{k+1}^i \leq x_{k+1}^{i-1}) = 1, \quad i = 2, 3. \quad (8)$$

The local control task is that each vehicle tracks its own reference signal r_k^i subject to the almost sure no-collision constraints, except for the first vehicle which has no constraints. The cost function is a conditional expectation and the reference signal in this problem replaces the target x^{i*} . For this coordination problem, the horizon $N = 1$ is sufficient and for simplicity, there is no constraint posed on the control u^i . Although the noise signals have uniform distribution rather than gaussian distribution, the ideas of (Yan and Bitmead, 2005) still apply to convert the probabilistic constraint (8) to deterministic.

Reference Trajectories: We consider two problem formulations which are distinguished by their reference specification. Define the fleet reference trajectory,

$$r_k^* = 250k, \quad 0 \leq k \leq 3,$$

where distance, r_k^* , is measured in kilometers and time, k , in hours. The two strategies are

- Leader-Follower Strategy: $r_k^1 = r_k^*$, $r_k^2 = x_k^1 - d_1$, $r_k^3 = x_k^2 - d_2$.
- Fixed-Reference Strategy: $r_k^1 = r_k^*$, $r_k^2 = r_k^1 - d_1 = r_k^* - d_1$, $r_k^3 = r_k^2 - d_2 = r_k^* - d_1 - d_2$.

Note that fixed-reference differs from the leader-follower strategy through the provision of absolute reference trajectories in place of relative positioning. This change will be shown to affect string stability through the achievement of decoupled control when constraints are inactive, as described in Theorem 3.

Performance indices: To evaluate these local controllers, the following performance indices are considered:

- String stability: we consider the effect of accumulating noises in the vehicle's closed-loop trajectory. The standard deviation of $\text{detrond}(x^i) = x^i - \text{trend}(x^i)$ is the index for string stability, where $\text{trend}(x^i)$ in this simulation is the best line fit of the resulting x^i . If this quantity is decreasing from Vehicle i to Vehicle $i + 1$ by a factor less than one, then the formation has string stability.

- Inter vehicle spacing: this implies efficiency and is preferred to be small in many circumstances. For example, in an automated highway system, the smaller distance between vehicles the higher is the efficiency of using the road.

4.2 Design Parameters and Simulation

This simple one-dimensional coordination problem exposes a number of design variables and, importantly for the purpose of this paper, their simple and conceptual interaction to achieve system performance. These design parameters are: control structure, examined in terms of leader-follower or fixed-reference design; information architecture, connected to communication link assignment, which is fixed here; target fleet structure, specified by separations d_1 and d_2 ; communication resource allocation, captured by noise bounds U and V .

As shown in the Appendix, the closed-loop states under leader-follower control are:

$$\begin{aligned} x_{k+1}^1 &= r_{k+1}^*, \\ x_{k+1}^2 &= r_{k+1}^* - \max\{d_1, \frac{U+V}{2}\} + (\nu_k^2 + \mu_k^2), \end{aligned} \tag{9}$$

$$\begin{aligned} x_{k+1}^3 &= x_{k+1}^2 - \max\{d_2, \frac{U+V}{2}\} + (\nu_k^3 + \mu_k^3) \\ &= r_{k+1}^* - \max\{d_1, \frac{U+V}{2}\} - \max\{d_2, \frac{U+V}{2}\} + (\nu_k^2 + \mu_k^2 + \nu_k^3 + \mu_k^3). \end{aligned} \tag{10}$$

Under fixed-reference control, the closed-loop states are:

$$\begin{aligned} x_{k+1}^1 &= r_{k+1}^*, \\ x_{k+1}^2 &= r_{k+1}^* - \max\{d_1, \frac{U+V}{2} - (\nu_k^2 + \mu_k^2)\} \end{aligned} \tag{11}$$

$$\begin{aligned} x_{k+1}^3 &= \min\{r_{k+1}^3, x_{k+1}^2 - \frac{U+V}{2} + (\nu_k^3 + \mu_k^3)\} \\ &= r_{k+1}^* - \max\{d_1 + d_2, \frac{U+V}{2} + \max\{d_1, \frac{U+V}{2} - (\nu_k^2 + \mu_k^2)\} - (\nu_k^3 + \mu_k^3)\}. \end{aligned} \tag{12}$$

For demonstration, we first consider two choices of design parameters to illustrate the achievable performance of these two strategies with fixed $U = V = 80$.

Case I: Minimum Distance Task. That is to keep the three trains as close as possible. The parameters should be $d_1 = d_2 = 0$ and the fixed references $r^2 = r^3 = r^1$. The resulting trajectories of the two strategies are given in

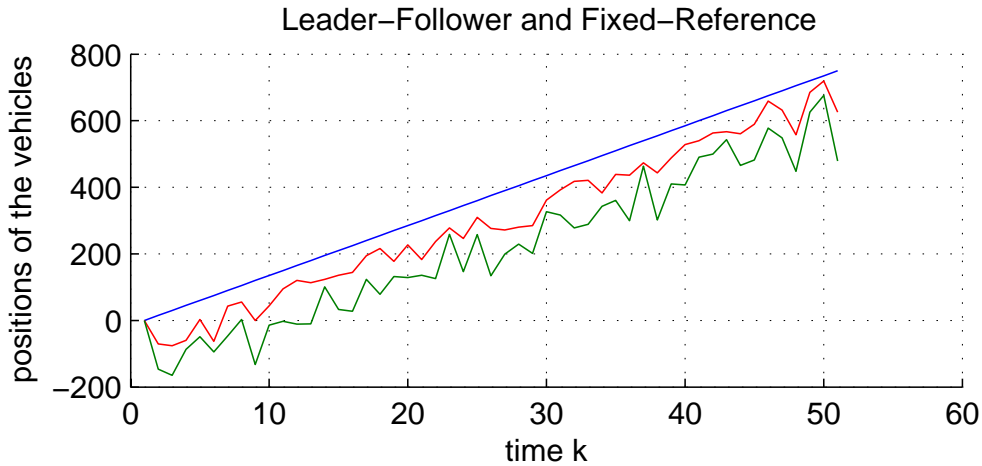


Fig. 1. Identical result of both strategies for the minimum distance task.

Figure 1. In this task, the two strategies give the same controller for each vehicle and hence their resulting plots are identical.

Remarks:

- Both strategies avoided collision. Vehicle 1 sits right on r^1 while Vehicle 2 and 3 try to stay close to Vehicle 1. However, due to the effect of the constraints, they are forced back to avoid collision.
- String instability occurs in both strategies. As shown in the figure, the trajectories of Vehicle 2 and Vehicle 3 look like random walks with a trend. We compute the standard deviation of $detrend(x^i)$ as an index of the string stability. The results are 0 for Vehicle 1, 31.1779 for Vehicle 2, and 51.2069 for Vehicle 3. For larger formations, this quantity will keep increasing and the resulting behavior of the later vehicles is unacceptable.

Case II: Fixed Separation Task. That is to have the three vehicles travelling a certain distance apart. The parameters are $d_1 = d_2 = 160$ and the fixed references $r^2 = r^1 - d_1$ and $r^3 = r^2 - d_2$. The resulting trajectories of the two strategies are given in Figure 2. As seen in the plots, string instability still exists in the leader-follower strategy

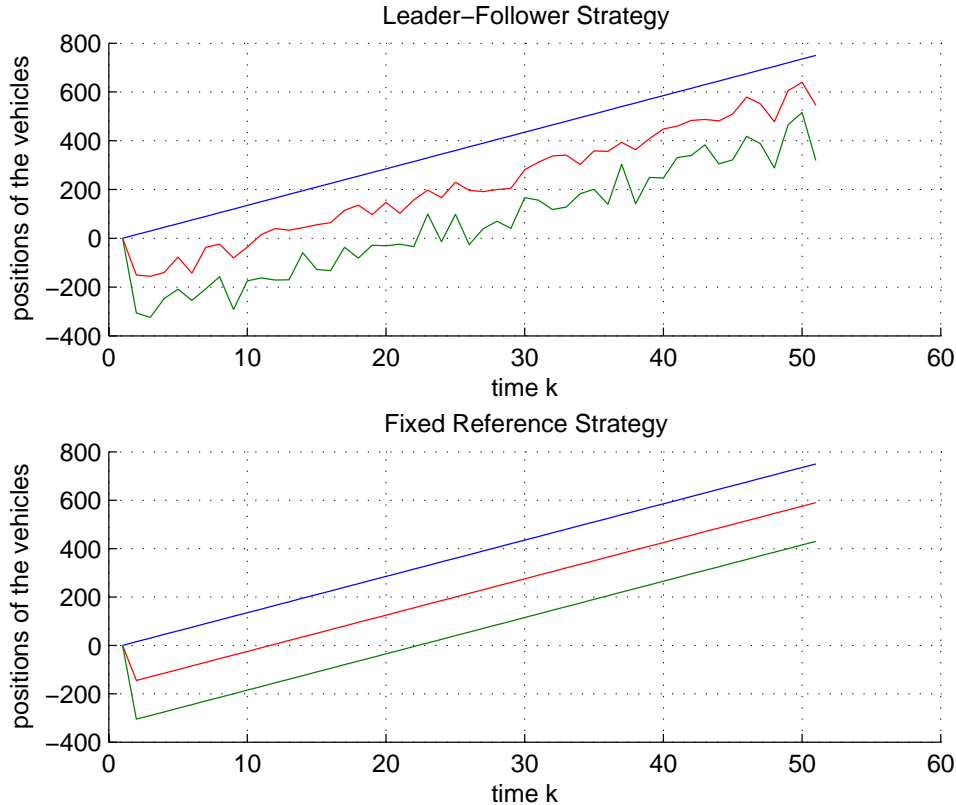


Fig. 2. Comparison of the two strategies in the fixed separation task with $d^i > U + V$.

with the same $detrend(x^i) = 0, 31.1779, 51.2069$ for Vehicle 1, 2, and 3. The fixed-reference strategy achieved string stability with $detrend(x^i) = 0, (i = 1, 2, 3)$.

In this framework, string instability occurs when the controls introduce estimates, and hence communication noises, into the closed-loop behavior. In the leader-follower strategy, the controls of Vehicle 2 and 3 always do so; while the fixed-reference strategy can achieve string stability when the constraints are inactive. This can be inferred from the closed-loop system equations given earlier. For leader-follower strategy, the closed-loop behaviors of Vehicles 2 and 3 are governed by (9) and (10) and the communication noises are added onto the Vehicles' trajectories no matter the choices of d_1 and d_2 . For fixed-reference strategy, the closed-loop behaviors of Vehicles 2 and 3 are governed by (11) and (12). By increasing d_1 and d_2 , the effect of string instability diminishes and when $d_1, d_2 \geq U + V$, the resulting closed-loop trajectories of Vehicles 2 and 3 are their assigned reference trajectories r_k^2 and r_k^3 .

The simulation results of the fixed-reference strategy in Case I and Case II showed that string stability and small inter-vehicle spacing cannot be achieved simultaneously. The reason is the large noise bounds U and V that correspond

to low-quality communications and low communication cost. In Case I, the zero inter-vehicle spacing and the large communication noises keep the modified no-collision constraint always active and string instability is always present. In Case II, we separate reference signals from each other in order to maintain the constraints inactive. Though string instability can be removed, the separations d_1 and d_2 are large. Thus, to achieve string stability and small separations simultaneously we must increase the communication accuracy and we consider the following case.

Case III: Fixed Separation Task with High-quality Communications. To increase the accuracy of the local information, we set $U = V = 10$ and $d_1 = d_2 = 20$. The result is shown in the Figure 3. The vehicles are able to stay close while string stability is achieved. This high performance is due to the high bandwidth communication.

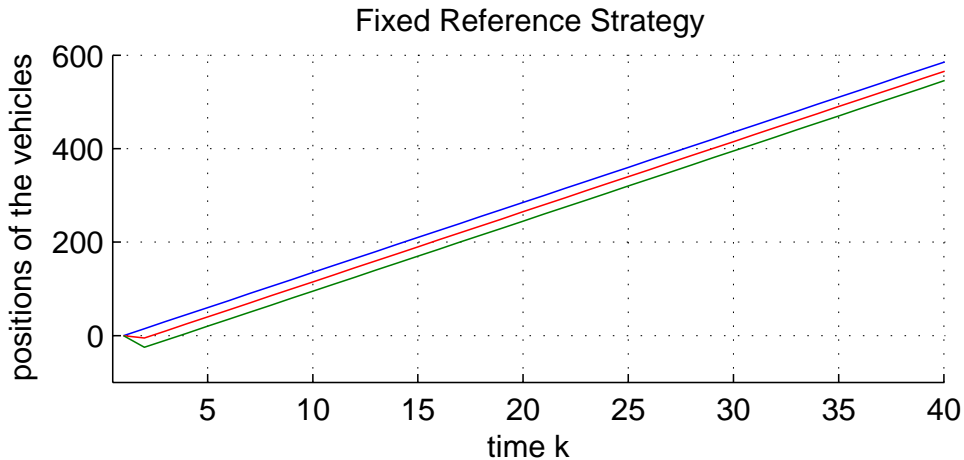


Fig. 3. Comparison of the two strategies in the fixed separation task.

4.2.1 Summary

- (1) In both minimum-distance and fixed-separation tasks, the constrained MPC approach in both strategies guarantees the no-collision requirement.
- (2) With the presence of disturbances, as stated in the Theorem 3, the fixed-reference strategy can completely remove string instability by choosing the proper references that deactivate the constraints. The closed-loop dynamics are decoupled. The determination of these proper references needs off-line work or online supervisor action.
- (3) The leader-follower strategy always display string instability with disturbance.
- (4) The inactivity of the constraints in the fixed-reference MPC relates the communication cost to the control performance.

5 Conclusion

In this paper, we examine the application of constrained model-predictive approach to a coordinated control problem. The direct inclusion of constraints makes the design easy and the global properties can be deduced from the local properties that are guaranteed by the local constrained MPC controllers.

We discussed the multi-vehicle coordination problem as a working example. The local constrained MPC controllers have fixed reference trajectories in their criterion functions and direct inclusion of no-collision constraints coupling the vehicles' dynamics. For deterministic problems, the asymptotic stability of each vehicle can be guaranteed via the local constrained MPC controller formulations and the global formation stability follows as summarized in Theorems 1 and 2. When stochastic disturbances are in the picture, the constraints of MPC should be modified to reflect the uncertainty so that collisions can be prevented. Theorem 3 provides a way to completely reject disturbance via decoupling the vehicles' closed-loop dynamics. As shown in the 1-D problem, the inclusion of fixed references and modified constraints is proven to be able to decouple the vehicles in their closed-loop dynamics via keeping the modified constraints inactive and thereby to achieve string stability. The relation revealed between the control

performance and the communication disturbance is a guide to design.

Appendix

A Fixed-Reference MPC Formulation and Solution

The local MPC problem for Vehicle 1:

$$\min_{u_k^1} J(x_k^1, u_k^1) = \min_{u_k^1} (x_{k+1}^1 - r_{k+1}^1)^2, \quad (\text{A.1})$$

with the solution $u_k^1 = -x_k^1 + r_{k+1}^1$.

The local MPC problem for Vehicle 2 or 3:

$$\left. \begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= \min_{u_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \\ \text{subject to: } P_{\mathcal{D}_k^i}(x_{k+1}^i \leq x_{k+1}^{i-1}) &= 1, \quad i = 2, 3, \end{aligned} \right\} \quad (\text{A.2})$$

To change (A.2) into a deterministic problem involving estimates, first we choose the unbiased estimates of the other vehicle at Vehicle i to be

$$\begin{aligned} \hat{x}_{k|k}^{i-1} &= [1 \ 0] z_k^i, \\ \hat{u}_k^{i-1} &= [0 \ 1] z_k^i, \\ \hat{x}_{k+1|k}^{i-1} &= \hat{x}_{k|k}^{i-1} + \hat{u}_k^{i-1} = [1 \ 1] z_k^i. \end{aligned}$$

The corresponding estimate errors are

$$\begin{aligned} \tilde{x}_{k|k}^{i-1} &= x_k^{i-1} - \hat{x}_{k|k}^{i-1} = -\nu_k^i, \\ \tilde{u}_{k|k}^{i-1} &= u_k^{i-1} - \hat{u}_{k|k}^{i-1} = -\mu_k^i, \\ \tilde{x}_{k+1|k}^{i-1} &= x_{k+1}^{i-1} - \hat{x}_{k+1|k}^{i-1} = -\nu_k^i - \mu_k^i. \end{aligned}$$

Since $\nu_k^i \sim U[-\frac{V}{2}, \frac{V}{2}]$, $\mu_k^i \sim U[-\frac{U}{2}, \frac{U}{2}]$, it follows that $\tilde{x}_{k+1|k}^{i-1} \in [-\frac{U+V}{2}, \frac{U+V}{2}]$. Hence:

$$P_{\mathcal{D}_k^i}(x_{k+1}^i \leq x_{k+1}^{i-1}) = 1, \Leftrightarrow x_{k+1}^i \leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}.$$

The $\frac{U+V}{2}$ term is the necessary stand-off due to the uncertainty introduced by using $\hat{x}_{k+1|k}^{i-1}$. Therefore, the local MPC problems (A.2) for Vehicles 2 and 3 become

$$\left. \begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= \min_{u_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \\ \text{subject to: } x_{k+1}^i &\leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}, \quad i = 2, 3, \end{aligned} \right\} \quad (\text{A.3})$$

Note that, in this transformation, the lower bounds of ν , μ , and $\tilde{x}_{k+1|k}^{i-1}$ are of importance.

The solution of (A.3) can be easily computed:

$$u_k^2 = \min \left\{ -x_k^2 + r_{k+1}^2, -x_k^2 + \hat{x}_{k+1|k}^1 - \frac{U+V}{2} \right\}, \quad (\text{A.4})$$

$$u_k^3 = \min \left\{ -x_k^3 + r_{k+1}^3, -x_k^3 + \hat{x}_{k+1|k}^2 - \frac{U+V}{2} \right\}. \quad (\text{A.5})$$

B Leader Follower MPC Formulation and Solution

The criterion function in the leader-follower MPC problem for Vehicle 2 and 3 should be changed to:

$$\begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= E_{\mathcal{X}_k^i} [x_{k+1}^i - (x_{k+1}^{i-1} - d_{i-1})]^2, \\ \text{subject to: } P_{\mathcal{X}_k^i}(x_{k+1}^i \leq x_{k+1}^{i-1}) &= 1, \quad i = 2, 3. \end{aligned}$$

The criterion function is posed as a conditional expectation due to the inaccuracy of x^{i-1} at Vehicle i . By the same process, this MPC problem is equivalent to:

$$\begin{aligned} \min_{u_k^i} J'(x_k^i, u_k^i) &= [x_{k+1}^i - (\hat{x}_{k+1|k}^{i-1} - d_{i-1})]^2, \\ \text{subject to: } x_{k+1}^i &\leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}, \quad i = 2, 3, \end{aligned}$$

and the solution is:

$$u_k^2 = \min \left\{ -x_k^2 + \hat{x}_{k+1|k}^1 - d_1, -x_k^2 + \hat{x}_{k+1|k}^1 - \frac{U+V}{2} \right\}, \quad (\text{B.1})$$

$$u_k^3 = \min \left\{ -x_k^3 + \hat{x}_{k+1|k}^2 - d_2, -x_k^3 + \hat{x}_{k+1|k}^2 - \frac{U+V}{2} \right\}. \quad (\text{B.2})$$

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