

Brief paper

Robust constrained predictive control using comparison model[☆]

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Abstract

This paper proposes a quadratic programming (QP) approach to robust model predictive control (MPC) for constrained linear systems having both model uncertainties and bounded disturbances. To this end, we construct an additional comparison model for worst-case analysis based on a robust control Lyapunov function (RCLF) for the unconstrained system (not necessarily an RCLF in the presence of constraints). This comparison model enables us to transform the given robust MPC problem into a nominal one without uncertain terms. Based on a terminal constraint obtained from the comparison model, we derive a condition for initial states under which the ultimate boundedness of the closed loop is guaranteed without violating state and control constraints. Since this terminal condition is described by linear constraints, the control optimization can be reduced to a QP problem.

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1. Introduction

Model predictive control (MPC), which determines online the control input by solving a finite horizon open-loop control optimization problem, is one of the few tractable ways to handle design problems having input and state constraints. Robust MPC, which explicitly takes account of model uncertainties, has been studied from the end of 1980s. (See Bemporad & Morari (1999) and Mayne, Rawling, Rao, & Scokaert (2000) for overviews.) This paper is particularly related to the state-space approaches (Lee & Yu, 1997; Michalska & Mayne, 1993; Scokaert & Mayne, 1998) rather than the early works based on impulse response models (Allwright & Papavasiliou, 1992; Campo & Morari, 1987; Zheng & Morari, 1993).

One of the main difficulties in robust MPC for practical use is the complexity of online optimal control problems. Because of uncertain terms in prediction models, the optimal control problem at each time step is typically described as a min–max optimization problem subject to the condition that given constraints are satisfied for all possible uncertainties. Therefore, it might be more practical to simplify such optimal control problems, even if we have to give up the strictly optimal solution.

Computationally efficient constrained MPC methods in the presence of model uncertainties have been proposed in Kothare, Rossiter and Schuurmans (1996) and Kouvaritakis, Balakrishnan and Morari (2000). A key assumption in these methods is that initial states belong to a robust invariant set given by a linear or an affine state feedback law satisfying given constraints inside the set. This assumption enables simplification of the optimal control problems in robust MPC, since prediction of future states is not required for ensuring robust stability. The price of this computational improvement might be that the admissible initial states are restricted to smaller sets compared with other approaches based on terminal conditions for predicted state trajectories.

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In many other MPC methods, robust stability is guaranteed if predicted states at the terminal time belong to a robustly invariant set for all possible uncertainties. Two types of methods, which require convex optimization instead of min–max optimization, have been proposed based on polyhedral invariant sets (Lee & Kouvaritakis, 1999, 2000, 2002) and ellipsoidal sets (Schuurmans & Rossiter, 2000), respectively. A merit of using ellipsoidal invariant sets is that they can be constructed more easily by using Lyapunov equations. An open problem of the existing method using ellipsoidal invariant sets is that the optimal control problem is still too complex, since the number of constraints increases rapidly with the prediction horizon. Another limitation is that it does not explicitly deal with disturbances.

In this paper, we propose a new robust MPC method based on ellipsoidal invariant sets for constrained linear systems with model uncertainties and disturbances. To predict the worst case of the model uncertainties with modest increase of constraints, we introduce a comparison model based on a robust control Lyapunov function (RCLF) for the unconstrained system, which is not necessarily an RCLF in the presence of constraints. This comparison model enables us to transform the given robust MPC problem to a nominal one without uncertain terms. Based on a terminal constraint obtained from the comparison model, we derive a condition for initial states under which the ultimate boundedness of the closed loop is guaranteed without violating state and control constraints. Furthermore, since this terminal condition is described as linear constraints, the control optimization problem for the proposed method can be solved by standard quadratic programming (QP) methods.

2. Problem formulation

Let $\|x\|$ and $\|x\|_p$ denote the Euclidean and p -norms of a vector x , respectively. The symbol x_i denotes the i th element of a vector x . Let $\bar{\sigma}(M)$ and $\|M\|_\infty$ denote the largest singular value and the induced ∞ -norm of a matrix M . For Hermitian matrices M , $\bar{\lambda}(M)$ and $\underline{\lambda}(M)$ denote the largest and smallest eigenvalues, respectively. The notation $M > 0$ means that M is symmetric positive definite, and $M^{1/2}$ denotes the unique positive definite square root of $M > 0$.

We consider linear systems:

$$\dot{x} = Ax + Bu + B_d d, \quad x(0) = x_0, \quad (1)$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $B_d \in R^{n \times p}$, and x_0 is a given initial state. The constraints for the state $x(t) \in R^n$ and the control $u(t) \in R^m$ are described as

$$\begin{aligned} x(t) \in X, \quad u(t) \in U \quad \forall t \geq 0, \\ X := \{x \in R^n : |x_i| \leq \gamma_i \quad \forall i\}, \\ U := \{u \in R^m : |u_i| \leq \eta_i \quad \forall i\} \end{aligned} \quad (2)$$

with given constant vectors $\gamma \in R^n$ and $\eta \in R^m$, while the disturbance $d(t) \in R^p$, which consists of a state-dependent

uncertainty $d_1(t) \in R^{p_1}$ and a bounded disturbance $d_2 \in R^{p_2}$, satisfies

$$\begin{aligned} d(t) &:= [d_1^T(t), d_2^T(t)]^T \in D(x(t)) \quad \forall t \geq 0, \\ D(x) &:= D_1(x) \times D_2, \\ D_1(x) &:= \{d_1 \in R^{p_1} : \|d_1\| \leq \|x\|\}, \\ D_2 &:= \{d_2 \in R^{p_2} : \|d_2\| \leq 1\}. \end{aligned} \quad (3)$$

Corresponding to d_1 and d_2 , matrices $B_{d_1} \in R^{n \times p_1}$ and $B_{d_2} \in R^{n \times p_2}$ are defined such that $B_d = [B_{d_1}, B_{d_2}]$.

We assume that a feedback gain $K \in R^{m \times n}$ and a terminal set

$$X_f = \{x \in R^n : V(x) \leq 1\} \quad (4)$$

are given for $V(x) := \sqrt{x^T P x}$ ($P > 0$), such that the following conditions are satisfied.

Assumption 1.

$$X_f \subset X, \quad Kx \in U \quad \forall x \in X_f.$$

Assumption 2.

$$0 \leq \sqrt{\bar{\lambda}(P)} \frac{\alpha_2}{\alpha_1} < 1, \quad \alpha_1 > 0,$$

where

$$\begin{aligned} \alpha_1 &:= \underline{\lambda}(Q) - 2\bar{\sigma}(PB_{d_1}), \quad \alpha_2 := 2\bar{\sigma}(PB_{d_2}), \\ Q &:= -(PA_c + A_c^T P), \quad A_c := A + BK. \end{aligned}$$

Assumption 1 states that the given feedback control $u = Kx$ always satisfies the constraints in X_f , whereas Assumption 2 implies

$$\sup_{d \in D(x)} \dot{V}(x) < 0 \quad \forall x \in X_f \setminus \Omega, \quad (5)$$

as verified in Section 3, where

$$\Omega := \left\{ x \in R^n : V(x) \leq \sqrt{\bar{\lambda}(P)} \frac{\alpha_2}{\alpha_1} \right\}.$$

Therefore, these assumptions, which are standard in robust MPC (Mayne, Rawling, Rao, & Scokaert, 2000), show that X_f is a robustly invariant set by using the feedback control $u = Kx$. Note that (5) shows that $V(x)$ would be a RCLF, if the system were not constrained.

Under these assumptions, our goal is to construct an MPC method which guarantees the ultimate boundedness of the closed loop without violating constraints for any $d(t) \in D(x(t))$ ($t > 0$). To avoid min–max optimization, our method does not directly use the uncertain model (1), (3) in the optimal control problem. Instead, we use a nominal model of (1) and an additional model, which is constructed based on the unconstrained RCLF $V(x)$ and a priori information in (3).

3. Proposed MPC method

As mentioned in Section 2, we predict the state $x(\tau)$ of (1) based on the nominal model which is as follows:

$$\dot{\hat{x}} = A\hat{x} + B\hat{u}, \quad \hat{x}(t|t) = x(t), \quad (6)$$

where $\hat{x}(\tau|t)$ denotes the predicted value of $x(\tau)$ at the current time t . For robustness analysis of the closed loop, it is necessary to estimate prediction error bounds of \hat{x} due to the disturbance term d in (1). The predicted control $\hat{u}(\tau|t)$ in (6) has the form

$$\hat{u} = K\hat{x} + \tilde{u}, \quad (7)$$

where $\tilde{u}(\tau|t)$ is an open-loop trajectory and K is the feedback gain given in Section 2. This type of control is often used in the literature to prevent the prediction error bounds from becoming too conservative. (See e.g. Bemporad (1998) and Lee & Kouvaritakis (1999, 2000).) By using (7), the real system (1) and the nominal model (6) are written as

$$\dot{x} = A_c x + B\tilde{u} + B_d d, \quad \dot{\hat{x}} = A_c \hat{x} + B\tilde{u}. \quad (8)$$

The first step to obtain a MPC method dealing with the problem in Section 3 is to derive constraints for \hat{x} and \hat{u} , which guarantee that the constraints in (2) for the real system are also satisfied. However, since the disturbance set D depends on the state x of the real system as in (3), it is difficult to evaluate the prediction error from only the nominal model and a priori information on uncertainties. The second step is to derive a terminal condition to ensure the ultimate boundedness of the closed loop. This is not an easy task even in the case of bounded disturbance only (i.e. $B_{d1} = 0$), since we use the nominal model (6) rather than (1).

To overcome these difficulties, we introduce the following scalar system constructed based on the unconstrained RCLF $V(x)$ in (5) and a priori information on uncertainties

$$\dot{w} = -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, \quad w(t|t) = V(x(t)),$$

$$a_1 := \frac{\alpha_1}{2\bar{\lambda}(P)}, \quad a_2 := \frac{\alpha_2}{2\sqrt{\bar{\lambda}(P)}}, \quad b_i := \|P^{1/2} B_i\|, \quad (9)$$

where B_i is the i th column of B . This system enables us to obtain an upper bound on the future value of $V(x)$, as shown in the following lemma. We refer to system (9) as the comparison model, since the lemma is proved based on the comparison principle (Miller & Michel, 1982), which is known as a tool to obtain bounds on solutions of differential equations without computing solutions themselves through the use of differential inequalities (Flett, 1980).

Lemma 3. For any $\tilde{u}(\tau|t)$ ($\tau \in [t, t+T]$), the states of the comparison system in (9) and the real system in (8) satisfy

$$V(x(\tau)) \leq w(\tau|t), \quad \tau \in [t, t+T]. \quad (10)$$

Proof. Since $V(x) = \sqrt{x^T P x}$, we have

$$\begin{aligned} \dot{V}(x) &= \frac{1}{2V} (\dot{x}^T P x + x^T P \dot{x}) \\ &= \frac{1}{2V} (x^T (A_c^T P + P A_c) x + 2x^T P (B_d d + B\tilde{u})) \\ &\leq -\frac{\|x\|}{2V} (\underline{\lambda}(Q)\|x\| - 2\bar{\sigma}(P B_{d1})\|d_1\| \\ &\quad - 2\bar{\sigma}(P B_{d2})\|d_2\|) \\ &\quad + \frac{1}{V} \|P^{1/2} x\| \|P^{1/2} B\tilde{u}\|. \end{aligned}$$

It follows from $d \in D(x)$ and $V(x) \leq \sqrt{\bar{\lambda}(P)}\|x\|$ that

$$\begin{aligned} \dot{V}(x) &\leq -\frac{\|x\|}{2V} (\alpha_1 \|x\| - \alpha_2) + \|P^{1/2} B\tilde{u}\| \\ &\leq -\frac{\alpha_1}{2\bar{\lambda}(P)} V + \frac{\alpha_2}{2\sqrt{\bar{\lambda}(P)}} + \sum_{i=1}^m \|P^{1/2} B_i\| |\tilde{u}_i| \\ &= -a_1 V + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|. \end{aligned} \quad (11)$$

Thus, (10) is shown by the comparison principle (Miller & Michel, 1982). \square

Note that property (5) is easily verified, since inequality (11) shows that, for $u = Kx$ ($\tilde{u} = 0$),

$$\dot{V}(x) \leq -a_1 \left(V - \frac{a_2}{a_1} \right) = -a_1 \left(V - \sqrt{\bar{\lambda}(P)} \frac{\alpha_2}{\alpha_1} \right). \quad (12)$$

Once we find the model for predicting an upper bound w on $V(x)$, constraint sets for \hat{x} and \hat{u} can be derived as follows:

$$\begin{aligned} \hat{X}(\tau, w(\cdot|t)) &:= \{x \in R^n : |x_j| \leq \gamma_j - \hat{\gamma}_j(\tau, w(\cdot|t)), \forall j\}, \\ \hat{U}(\tau, w(\cdot|t)) &:= \{u \in R^m : |u_i| \leq \eta_i - \hat{\eta}_i(\tau, w(\cdot|t)), \forall i\}, \end{aligned}$$

where

$$\begin{aligned} \hat{\gamma}_i(\tau, w(\cdot|t)) &:= \int_0^{\tau-t} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\bar{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \\ \hat{\eta}_i(\tau, w(\cdot|t)) &:= \int_0^{\tau-t} \left(\|\xi_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\bar{\lambda}(P)}} + \|\xi_{2i}^T(s)\|_1 \right) ds \end{aligned} \quad (13)$$

and $\zeta_{ji}(t)$, $\xi_{ji}(t)$ ($j = 1, 2$) denote the i th rows of $\zeta_j(t) := e^{A_c t} B_{dj}$, $\xi_j(t) := K e^{A_c t} B_{dj}$. Similar modification of constraints is considered in Chisci, Rossiter and Zappa (2001), Mayne and Langson (2001) and Michalska and Mayne (1993) in the case of bounded disturbance only (i.e. $B_{d1} = 0$).

The constraints based on \hat{X} and \hat{U} have the following property.

Theorem 4. For a given $x(t) \in R^n$, any open-loop trajectory $\tilde{u}(\tau|t)$, $\tau \in [t, t+T]$, which satisfies

$$\begin{aligned} \dot{\hat{x}} &= A_c \hat{x} + B \tilde{u}, \quad \hat{x}(t|t) = x(t), \\ \dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, \quad w(t|t) = V(x(t)), \\ \hat{x}(\tau|t) &\in \hat{X}(\tau, w(\cdot|t)), \\ \tilde{u}(\tau|t) + K \hat{x}(\tau|t) &\in \hat{U}(\tau, w(\cdot|t)) \end{aligned} \quad (14)$$

also satisfies the constraints for the real system (1)

$$x(\tau) \in X, \quad \tilde{u}(\tau|t) + Kx(\tau) \in U \quad (15)$$

for all possible $d(\tau) \in D(x(\tau))$, $\tau \in [t, t+T]$.

Proof. From the differential equations in (8), we have

$$\hat{x}(\tau|t) = e^{A_c(\tau-t)} x(t) + \int_0^{\tau-t} e^{A_c s} B \tilde{u}(\tau-s|t) ds, \quad (16)$$

$$\begin{aligned} x(\tau) &= e^{A_c(\tau-t)} x(t) + \int_0^{\tau-t} e^{A_c s} B \tilde{u}(\tau-s|t) ds \\ &\quad + \int_0^{\tau-t} e^{A_c s} B_d d(\tau-s) ds. \end{aligned} \quad (17)$$

It follows from $d(\tau) \in D(x(\tau))$ and Lemma 3 that

$$\begin{aligned} |x_i(\tau) - \hat{x}_i(\tau|t)| &\leq \int_0^{\tau-t} (|\zeta_{1i}^T(s) d_1(\tau-s)| + |\zeta_{2i}^T(s) d_2(\tau-s)|) ds \\ &\leq \int_0^{\tau-t} (\|\zeta_{1i}^T(s)\|_1 \|d_1(\tau-s)\|_\infty \\ &\quad + \|\zeta_{2i}^T(s)\|_1 \|d_2(\tau-s)\|_\infty) ds \\ &\leq \int_0^{\tau-t} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\lambda(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds. \end{aligned} \quad (18)$$

Therefore, from (13) and (18),

$$\begin{aligned} |x_i(\tau)| &\leq |\hat{x}_i(\tau|t)| + |x_i(\tau) - \hat{x}_i(\tau|t)| \\ &\leq |\hat{x}_i(\tau|t)| + \hat{\gamma}_i(\tau, w(\cdot|t)). \end{aligned}$$

This implies that each $\tilde{u}(\tau|t)$, which satisfies $\hat{x}(\tau|t) \in \hat{X}(\tau, w(\cdot|t))$ in (14), also satisfies $x(\tau) \in X$ in (15) for all $d(\tau) \in D(x(\tau))$, $\tau \in [t, t+T]$. Similarly to (18), we have

$$\begin{aligned} |u_i(\tau) - \hat{u}_i(\tau|t)| &\leq \int_0^{\tau-t} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\lambda(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \end{aligned}$$

from (16) and (17). Therefore, any $\tilde{u}(\tau|t)$ satisfying (14) also satisfies (15) for all $d(\tau) \in D(x(\tau))$, $\tau \in [t, t+T]$. \square

Based on Theorem 4, the optimization problem for the proposed MPC method is described for a given diagonal matrix $R > 0$ as follows:

Proposed MPC:

$$\begin{aligned} \min_{\tilde{u}} J(x(t), \tilde{u}(\cdot|t)) &:= \int_{\tau=t}^{t+T} \tilde{u}(\tau|t)^T R \tilde{u}(\tau|t) d\tau \\ \text{subject to (14) and} \\ w(\tau|t) &\leq \omega, \quad \tau \in [t, t+T] \\ w(t+T|t) &\leq 1. \end{aligned} \quad (19)$$

In (19), the finite-horizon optimal control problem without any uncertain parameters is solved from the measured state $x(t)$ at the current time t . The optimal control trajectory $\tilde{u}^*(\tau|t)$ of (19) is implemented until the next update time $t+\delta$. It is easily seen from Theorem 4 that, if the problem in (19) is feasible at each update time, then the given constraints are always satisfied.

In order to guarantee the feasibility at each time and the ultimate boundedness of the closed loop, we need the constraints for w in (19). It can be seen from (4) and (10) that the terminal condition $(w(t+T) \leq 1)$ guarantees $x(t+T) \in X_f$ for the real system. The constant ω is a number satisfying $\omega \geq \max\{V(x_0), 1\}$. In Section 4, an additional assumption, which gives an upper bound on ω , is given for ensuring feasibility and ultimate boundedness. The cost functions J of optimal control problems often play important roles in stability analysis of MPC methods. In the proposed method, we choose the same cost function as in Lee and Kouvaritakis (1999, 2000), which is probably the simplest one to ensure robust stability. Robust analysis for other types of cost functions is one of the future issues.

Remark 5. The w -differential equation in (14), which is a constraint in (19), is nonlinear in \tilde{u}_i . By introducing a new variable $v(\tau|t) \in R^m$ and using the property that R is diagonal, we modify the constraint to

$$\begin{aligned} \dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i v_i, \quad w(t|t) = V(x(t)) \\ |\tilde{u}_i(\tau|t)| &\leq v_i(\tau|t), \quad \tau \in [t, t+T], \quad i = 1, \dots, m \end{aligned}$$

and the cost function $J(x(t), \tilde{u}(\cdot|t))$ to $J(x(t), v(\cdot|t))$. Therefore, the modified problem has only linear constraints and gives the same solution as (19), since the optimal solution of the modified problem always satisfies $|\tilde{u}_i(\tau|t)| = v_i(\tau|t)$. The recast problem with the linear constraints and the quadratic cost function has free variables $\tilde{u}_i(\tau|t)$ and $v_i(\tau|t)$.

Remark 6. In (19), the predicted nominal state \hat{x} decays faster than w , since w is an upper bound on $V(x)$. Therefore, for a control horizon $T_u < T$ long enough for \hat{x} to converge to X_f , the computational burden can be decreased by applying \tilde{u} of the following form:

$$\tilde{u}(\tau|t) = \begin{cases} \tilde{u}^*(\tau|t) & \text{if } t \leq \tau \leq t + T_u, \\ 0 & \text{if } \tau > t + T_u. \end{cases} \quad (20)$$

4. Feasibility and stability results

As mentioned in the previous section, we need the following assumption prescribing upper bounds on ω to ensure that the optimal control problem in (19) is feasible at each time.

Assumption 1'. The given $\omega (\geq \max\{V(x_0), 1\})$ in (19) satisfies

$$\begin{aligned} \omega \|\zeta_1\|_{\mathcal{L}_1(T)} &\leq \sqrt{\underline{\lambda}(P)} \left(\min_i \gamma_i - \|\zeta_2\|_{\mathcal{L}_1(T)} \right) - 1 \\ \omega \|\xi_1\|_{\mathcal{L}_1(T)} &\leq \sqrt{\underline{\lambda}(P)} \left(\min_i \eta_i - \|\xi_2\|_{\mathcal{L}_1(T)} \right) \\ &\quad - \max_i \|K_i^T\|, \end{aligned}$$

where K_i denotes the i th row of K and $\|\zeta_1\|_{\mathcal{L}_1(T)} := \int_0^T \|\zeta_1(s)\|_\infty ds$.

As verified in the last part of the Appendix, Assumption 1' is given such that the same type of condition as Assumption 1 is satisfied for the modified constraint sets \hat{X} and \hat{U} . That is

$$\begin{aligned} X_f &\subset \hat{X}(\tau, w(\cdot|t)) \subseteq X \\ Kx &\in \hat{U}(\tau, w(\cdot|t)) \subseteq U \quad \forall x \in X_f. \end{aligned} \quad (21)$$

Therefore, Assumption 1' is a sufficient condition for Assumption 1. If Assumption 1' cannot be satisfied for any $\omega > \max\{V(x_0), 1\}$, we need to consider a smaller terminal set X_f or to modify the feedback gain K to satisfy Assumption 1'. Note that, in the case where the control, as in (20), is adopted, a weaker condition than Assumption 1' can be obtained by modifying the left-hand sides in the inequalities.

The following theorem describes the properties of the feasibility and the ultimate boundedness of the proposed MPC method.

Theorem 7. Assume that the optimization in (19) is feasible at $t = 0$ for ω which satisfies Assumption 1'. Then, the proposed MPC method has the following properties:

- (i) the optimization in (19) is feasible at each $t > 0$,
- (ii) for any $\mu > a_2/a_1$, there exists t_c such that

$$\|x\| \leq \frac{\mu}{\sqrt{\underline{\lambda}(P)}} \quad \forall t \geq t_c. \quad (22)$$

To prove Theorem 7, we need the following key result.

Lemma 8. Assume the optimization problem in (19) is feasible at the current time t for ω which satisfies Assumption 1'. Then, at the next time step $t + \delta$,

$$\tilde{u}(\tau|t + \delta) = \begin{cases} \tilde{u}^*(\tau|t), & \tau \in [t + \delta, t + T], \\ 0, & \tau \in [t + T, t + T + \delta] \end{cases} \quad (23)$$

is a feasible solution of (19), where $\tilde{u}^*(\tau|t)$ denotes the optimal solution at t .

Proof. See Appendix. \square

Proof of Theorem 7. We show (i) by induction. The optimization problem is feasible at $t = 0$ by the assumption. Assume now it is feasible at each $t = i\delta (i = 1, \dots, k)$. Then, since Lemma 8 shows that the control in (23) is feasible at $t = (k + 1)\delta$, (i) is proved.

In order to prove (ii), we next show that the optimal cost $J(x(t), \tilde{u}^*)$ is nonincreasing. At the time step $t + \delta$, the feasible solution in (23) satisfies

$$J(x(t + \delta), \tilde{u}(\cdot|t + \delta)) \leq J(x(t), \tilde{u}^*(\cdot|t)), \quad (24)$$

since

$$\begin{aligned} &J(x(t + \delta), \tilde{u}(\cdot|t + \delta)) - J(x(t), \tilde{u}^*(\cdot|t)) \\ &= \int_{t+\delta}^{t+\delta+T} \tilde{u}(\tau|t + \delta)^T R \tilde{u}(\tau|t + \delta) d\tau \\ &\quad - \int_t^{t+T} \tilde{u}^*(\tau|t)^T R \tilde{u}^*(\tau|t) d\tau \\ &= - \int_{\tau=t}^{t+\delta} \tilde{u}^*(\tau|t)^T R \tilde{u}^*(\tau|t) d\tau \leq 0. \end{aligned} \quad (25)$$

It is also satisfied that

$$J(x(t + \delta), \tilde{u}^*(\cdot|t + \delta)) \leq J(x(t + \delta), \tilde{u}(\cdot|t + \delta)), \quad (26)$$

from the optimality of $J(x(t + \delta), \tilde{u}^*(\cdot|t + \delta))$. From (24) and (26), the optimal cost is nonincreasing, that is

$$J(x(t + \delta), \tilde{u}^*(\cdot|t + \delta)) \leq J(x(t), \tilde{u}^*(\cdot|t)).$$

Since the optimal cost is nonincreasing and bounded by 0 from below, it satisfies $J(x(t), \tilde{u}^*(\cdot|t)) \rightarrow c$ as $t \rightarrow \infty$ for a constant $c \geq 0$. This implies that, for each $\varepsilon > 0$, there exists $t_1 > 0$ such that

$$\begin{aligned} 0 \leq J(x(t), \tilde{u}^*(\cdot|t)) - J(x(t + \delta), \tilde{u}^*(\cdot|t + \delta)) &< \varepsilon, \\ \forall t \geq t_1. \end{aligned} \quad (27)$$

But, from (24)–(26), we have

$$\begin{aligned} &\int_t^{t+\delta} \tilde{u}^*(\tau|t)^T R \tilde{u}^*(\tau|t) d\tau \\ &= J(x(t), \tilde{u}^*(\cdot|t)) - J(x(t + \delta), \tilde{u}(\cdot|t + \delta)) \\ &\leq J(x(t), \tilde{u}^*(\cdot|t)) - J(x(t + \delta), \tilde{u}^*(\cdot|t + \delta)). \end{aligned} \quad (28)$$

Thus, (27) and (28) imply

$$\tilde{u}^*(t|t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (29)$$

Since $a_2/a_1 < \mu$, given an $\varepsilon_1 > 0$ which satisfies

$$\frac{a_2 + \varepsilon_1 \sum_{i=1}^m b_i}{a_1} < \mu, \quad (30)$$

from (29) we can choose t_1 such that

$$\|\tilde{u}^*(t|t)\|_\infty \leq \varepsilon_1 \quad \forall t \geq t_1. \quad (31)$$

From (11) and (31), it follows that

$$\begin{aligned}\dot{V}(x(t)) &\leq -a_1 V(x(t)) + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i^*| \\ &\leq -a_1 V(x(t)) + a_2' \quad \forall t \geq t_1,\end{aligned}$$

where $a_2' := a_2 + \varepsilon_1 \sum_{i=1}^m b_i$. Therefore, by the comparison principle (Miller & Michel, 1982),

$$\begin{aligned}V(x(\tau)) &\leq e^{-a_1(\tau-t_1)} V(x(t_1)) + a_2' \int_0^{\tau-t_1} e^{-a_1 s} ds \\ &= e^{-a_1(\tau-t_1)} \left(V(x(t_1)) - \frac{a_2'}{a_1} \right) + \frac{a_2'}{a_1},\end{aligned}\quad (32)$$

and the right-hand side of (32) converges to a_2'/a_1 as $\tau \rightarrow \infty$. We first consider the case where $V(x(t_1)) > \mu$. Since (30) implies $a_2'/a_1 < \mu$, there exists a finite time $t_c (> t_1)$ which satisfies

$$e^{-a_1(\tau-t_1)} \left(V(x(t_1)) - \frac{a_2'}{a_1} \right) + \frac{a_2'}{a_1} = \mu.$$

Therefore,

$$V(x(\tau)) \leq \mu \quad \forall \tau \geq t_c. \quad (33)$$

On the other hand, if $V(x(t_1)) \leq \mu$, we have (33) for $t_c = t_1$, since (32) and (30) show that $V(x(\tau))$ cannot be greater than μ at any $t \geq t_1$. Thus (22) follows from (33). \square

Theorem 7 tells us that, if the MPC problem is feasible at $t = 0$, the state $x(t)$ converges into a ball around the origin with radius of $a_2/a_1 \sqrt{\lambda(P)}$. Particularly, if the uncertain term d_2 is zero, the state is steered to the origin. It is also important to notice that, from (29) and (33), the control law and the state converge to $u = Kx$ and Ω , respectively. Once the state is steered into the robustly invariant set X_f , the control law is completely switched to the feedback law $u = Kx$, since it is the optimal control in X_f in terms of the cost function (19).

5. Discrete-time case

In this section, we consider discrete-time systems

$$x(k+1) = Ax(k) + Bu(k) + B_d d(k), \quad x(0) = x_0. \quad (34)$$

The state and control constraints are described as

$$x(k) \in X, \quad u(k) \in U, \quad k = 0, 1, 2, \dots,$$

for X and U defined in (2), and the disturbance satisfies

$$d(k) \in D(x(k)), \quad k = 0, 1, 2, \dots$$

for $D(x)$ defined in (3). We assume that a given feedback gain $K \in R^{m \times n}$ and a terminal set X_f satisfy Assumption 1 and the following assumption modified from Assumption 2 for discrete-time systems.

Assumption 9.

$$0 \leq \frac{a_2}{1-a_1} < 1, \quad a_1 < 1,$$

where

$$a_1 := \sqrt{1 - \frac{\lambda(Q)}{\lambda(P)} + \frac{\bar{\sigma}(P^{1/2} B_{d_1})}{\sqrt{\lambda(P)}}}, \quad a_2 := \bar{\sigma}(P^{1/2} B_{d_2})$$

$$Q := P - A_c^T P A_c, \quad A_c := A + BK.$$

Under these assumptions, we derive a discrete-time MPC method based on the closed-loop prediction of the form in (7). In the same way as (8), the system in (34) is rewritten as

$$x(k+1) = A_c x(k) + B \tilde{u}(k) + B_d d(k), \quad x(0) = x_0. \quad (35)$$

We introduce the following comparison model:

$$\begin{aligned}w(\tau+1) &= a_1 w(\tau) + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i(\tau)|, \\ w(k) &= V(x(k)), \quad \tau = k, \dots, k+N-1,\end{aligned}\quad (36)$$

where $b_i = \|P^{1/2} B_i\|$ and B_i denotes the i th column of B . The comparison system (36) has the following property corresponding to Lemma 3:

Lemma 10. For any $\tilde{u}(\tau)$ ($\tau = k, \dots, k+N-1$), the comparison system in (36) and the real system in (35) satisfy

$$V(x(\tau)) \leq w(\tau), \quad \tau = k, \dots, k+N-1. \quad (37)$$

Proof. For $\bar{x} := A_c x + B \tilde{u} + B_d d$, we have

$$\begin{aligned}V(\bar{x}) &= \sqrt{\bar{x}^T P \bar{x}} = \|P^{1/2} (A_c x + B \tilde{u} + B_d d)\| \\ &\leq \|P^{1/2} A_c x\| + \|P^{1/2} B_d d\| + \|P^{1/2} B \tilde{u}\|.\end{aligned}\quad (38)$$

The first term in (38) satisfies

$$\begin{aligned}\|P^{1/2} A_c x\| &= \sqrt{x^T A_c^T P A_c x} = \sqrt{x^T (P - Q) x} \\ &= \sqrt{V^2(x) - x^T Q x}.\end{aligned}\quad (39)$$

Thus it follows from (38) and (39) that

$$\begin{aligned}V(\bar{x}) &\leq V(x) \sqrt{1 - \frac{\lambda(Q)}{\lambda(P)}} + \bar{\sigma}(P^{1/2} B_{d_1}) \|x\| \\ &\quad + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i| \\ &\leq a_1 V(x) + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|.\end{aligned}\quad (40)$$

Properties (36) and (40) imply (37) by induction. \square

Similar to the continuous-time case, the following constraint sets \hat{U} , \hat{X} depending on w are used to describe the proposed method.

$$\begin{aligned}\hat{X}(\tau, w(\cdot|k)) &= \{x \in R^n : |x_i| \leq \gamma_i - \hat{\gamma}_i(\tau, w(\cdot|k)) \forall i\}, \\ \hat{U}(\tau, w(\cdot|k)) &= \{u \in R^m : |u_i| \leq \eta_i - \hat{\eta}_i(\tau, w(\cdot|k)) \forall i\},\end{aligned}$$

where

$$\hat{\gamma}_i(\tau, w(\cdot|k)) := \sum_{s=0}^{\tau-k} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|k)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right)$$

$$\hat{\eta}_i(\tau, w(\cdot|k)) := \sum_{s=0}^{\tau-k} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|k)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right)$$

and $\zeta_{ji}(t)$, $\zeta_{ji}(t)$ ($j = 1, 2$) denote the i th rows of

$$\zeta_j(s) := \begin{cases} A_c^{s-1} B_{d_j} & \text{for } s \neq 0, \\ 0 & \text{for } s = 0, \end{cases} \quad \zeta_j(s) := K \zeta_j(s).$$

The optimization problem for the discrete-time MPC method is now described as follows:

Discrete-time MPC:

$$\min_{\tilde{u}} J(x(k), \tilde{u}(\cdot|k)) := \sum_{\tau=k}^{k+N-1} \tilde{u}(\tau|k)^T R \tilde{u}(\tau|k)$$

subject to

$$\hat{x}(\tau+1|k) = A_c \hat{x}(\tau|k) + B \tilde{u}(\tau|k), \quad \hat{x}(k|k) = x(k)$$

$$w(\tau+1|k) = a_1 w(\tau|k) + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i(\tau|k)|,$$

$$w(k|k) = V(x(k)),$$

$$\hat{x}(\tau+1|k) \in \hat{X}(\tau+1, w(\cdot|k))$$

$$\tilde{u}(\tau|k) + K \hat{x}(\tau|k) \in \hat{U}(\tau, w(\cdot|k))$$

$$w(\tau+1|k) \leq \omega, \quad w(k+N|k) \leq 1. \tag{41}$$

For the discrete-time MPC method, the following results can be derived in the same way as Theorem 7.

Theorem 11. Assume the optimization in (41) is feasible at $k = 0$ for $\omega (> \max\{V(x_0), 1\})$ satisfying

$$\omega \|\zeta_1\|_{\ell_1(T)} \leq \sqrt{\underline{\lambda}(P)} \left(\min_i \gamma_i - \|\zeta_2\|_{\ell_1(T)} \right) - 1$$

$$\omega \|\zeta_1\|_{\ell_1(T)} \leq \sqrt{\underline{\lambda}(P)} \left(\min_i \eta_i - \|\zeta_2\|_{\ell_1(T)} \right) - \max_i \|K_i^T\|,$$

where $\|\zeta_1\|_{\ell_1(N)} := \sum_{s=1}^N \|\zeta_1(s)\|_\infty$. Then, the proposed MPC method has the following properties:

- (i) the optimization in (41) is feasible at each $k > 0$,
- (ii) for any $\mu > a_2/(1 - a_1)$, there exists k_c such that

$$\|x(k)\| \leq \frac{\mu}{\sqrt{\underline{\lambda}(P)}}, \quad \forall k \geq k_c.$$

6. Numerical example

Consider the following uncertain system:

$$\dot{x} = \begin{bmatrix} 0 & 1 + \alpha \\ 1 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \beta, \tag{42}$$

where the state and control constraints are given as $|u(t)| \leq 2$, $|x_i(t)| \leq 1 (i = 1, 2)$ and the bounds on the uncertain

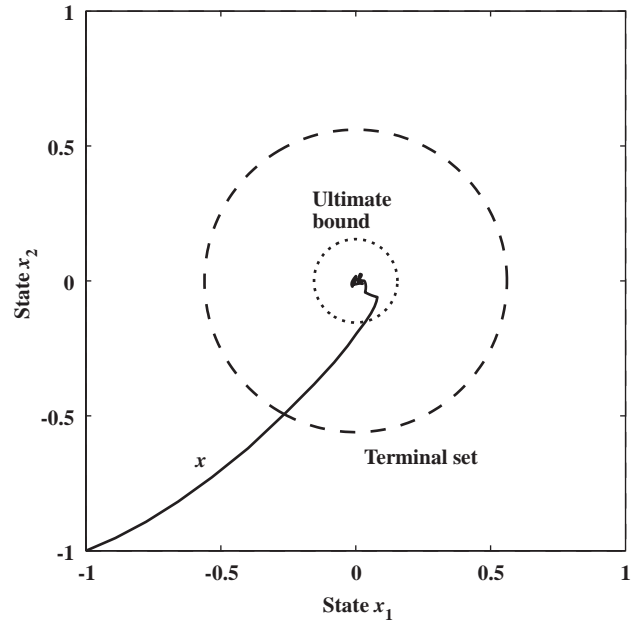


Fig. 1. Trajectory of the state.

parameters $|\alpha(t)| \leq 0.1$, $\|\beta(t)\| \leq 0.1$ are given as a priori information. The uncertain system in (42) is described as the form in (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_{d1} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},$$

$$B_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

A feedback gain K and a matrix P for the Lyapunov function, which satisfy Assumption 1' and 2, are chosen as

$$K = [-1.66, -2.32], \quad P = 2.33I. \tag{43}$$

In this case, the comparison system (9) is obtained as

$$\dot{w} = -0.647w + 0.1 + 1.41|\tilde{u}|. \tag{44}$$

We choose the horizons as $T_u = 1.0$, $T = 3.0$ s and the upper bound of w as $\omega = 2$, such that Assumption 1' is satisfied.

The proposed method is now applied to the systems in (42) and (44) discretized with sampling time 0.1 s. We show the simulation result for $\alpha = -0.1$ and $\beta(t)$ chosen as a step signal changing its value randomly every 0.5 s. The trajectory for 10 s of the state starting from $x_0 = [-1, -1]$ is shown by the solid line in Fig. 1. The dashed and dotted lines show the terminal set and the ultimate bound, respectively. From Fig. 1, it can be seen that the trajectory of the state goes into the terminal set and achieves ultimate boundedness against the uncertainties. In Fig. 2, the solid line shows the applied control trajectory $u(t)$, and the dash-dot line shows the predicted trajectory $\hat{u}(\tau|t)$ at $t = 0$, which are obtained by the proposed method. The dashed line shows the control trajectory by the given feedback controller in (43). As shown in Fig. 2, the given feedback controller violates the constraint,

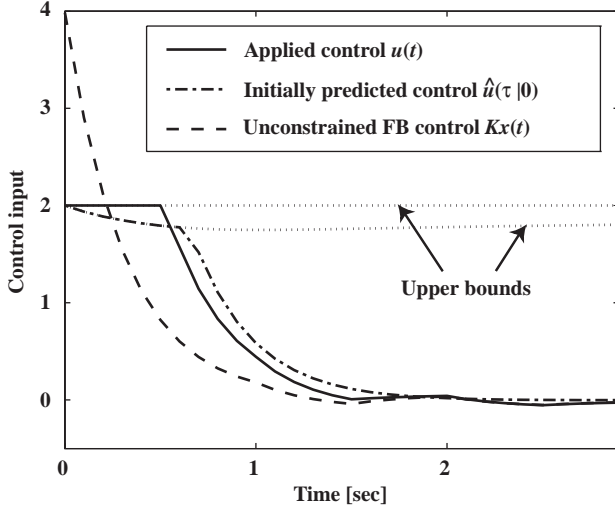


Fig. 2. Time plots of the controls.

since it is designed without taking account of the constraints. On the other hand, the solid line in Fig. 2 shows that the control obtained by the proposed MPC satisfies the given constraint. Note that, while the predicted trajectory \hat{u} at $t=0$ is chosen conservatively by taking account of the prediction error, the applied control can take the maximum allowable value $u(t) = 2$ at $t > 0$ by updating \hat{u} at each time step.

7. Conclusion

In this paper, we have proposed a new robust MPC method for constrained linear uncertain systems. The merits of the proposed method are summarized as (i) the control optimization is reduced to a QP rather than a min–max problem with modest increase of constraints, (ii) state-dependent uncertainties can be handled as well as bounded disturbances, and (iii) a condition for robust feasibility and the ultimate boundedness of the closed-loop is clarified. In order to obtain these properties, we have introduced an additional comparison model for worst-case analysis based on an RCLF for the unconstrained system. By using the comparison model, we have transformed the given robust MPC problem to a nominal one without uncertain terms. We have also shown that the terminal condition based on the comparison model ensures the feasibility and ultimate boundedness of the proposed method. Moreover, it has been shown that, since the terminal condition is described as linear constraints, the control optimization can be reduced to a QP problem.

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Appendix A. Proof of Lemma 8

In order to prove Lemma 8, we use the following two facts.

Lemma A.1. For given trajectories $\tilde{u}^*(\tau|t)$ ($\tau \in [t, t+T]$) and

$$\tilde{u}(\tau|t+\delta) = \tilde{u}^*(\tau|t), \quad \tau \in [t+\delta, t+T], \quad (\text{A.1})$$

the scalar system in (9) satisfies

$$w(\tau|t+\delta) \leq w(\tau|t), \quad \tau \in [t+\delta, t+T]. \quad (\text{A.2})$$

Proof. From (A.1) and (9), we have

$$\frac{d}{d\tau}(w(\tau|t+\delta) - w(\tau|t)) = -a_1(w(\tau|t+\delta) - w(\tau|t)).$$

Thus, it follows that

$$w(\tau|t+\delta) - w(\tau|t) = e^{-a_1(\tau-t-\delta)}(w(t+\delta|t+\delta) - w(t+\delta|t)). \quad (\text{A.3})$$

Since $w(t+\delta|t+\delta) = V(x(t+\delta)) \leq w(t+\delta|t)$ from Lemma 3, it follows from (A.3)

$$w(\tau|t+\delta) - w(\tau|t) \leq 0, \quad \tau \in [t+\delta, t+T]. \quad \square$$

Lemma A.2. Assume a predicted trajectory $w(\cdot|t)$ in (9) satisfies

$$w(\tau|t) \leq \omega, \quad \tau \in [t, t+T] \quad (\text{A.4})$$

for $\tilde{u}(\tau|t)$ at the current time t . Then, for the input $\tilde{u}(\tau|t+\delta)$ ($\tau \in [t+\delta, t+T]$) in (23), we have

$$\underline{\gamma} \leq \gamma_i - \hat{\gamma}_i(\tau, w(\cdot|t+\delta)), \quad (\text{A.5})$$

$$\underline{\eta} \leq \eta_i - \hat{\eta}_i(\tau, w(\cdot|t+\delta)) \quad (\text{A.6})$$

at the next time step $t+\delta$, where

$$\underline{\gamma} := \min_i \gamma_i - \frac{\omega}{\sqrt{\underline{\lambda}(P)}} \|\zeta_1\|_{\mathcal{L}_1(T)} - \|\zeta_2\|_{\mathcal{L}_1(T)}$$

$$\underline{\eta} := \min_i \eta_i - \frac{\omega}{\sqrt{\underline{\lambda}(P)}} \|\xi_1\|_{\mathcal{L}_1(T)} - \|\xi_2\|_{\mathcal{L}_1(T)}.$$

Proof. From (A.4) and the definition of $\hat{\gamma}_i$ in (13),

$$\begin{aligned} & \min_i \gamma_i + \hat{\gamma}_i(\tau, w(\cdot|t)) \\ & \leq \gamma_i + \frac{\omega}{\sqrt{\underline{\lambda}(P)}} \|\zeta_1\|_{\mathcal{L}_1(T)} + \|\zeta_2\|_{\mathcal{L}_1(T)}, \end{aligned}$$

which implies

$$\underline{\gamma} \leq \gamma_i - \hat{\gamma}_i(\tau, w(\cdot|t)). \quad (\text{A.7})$$

Also, from (13) and Lemma A.1, it follows that

$$\hat{\gamma}_i(\tau, w(\cdot|t+\delta)) \leq \hat{\gamma}_i(\tau, w(\cdot|t)). \quad (\text{A.8})$$

Thus, (A.5) is proved by (A.7) and (A.8). (A.6) follows similarly. \square

Based on the results above, we first prove the feasibility of $\tilde{u}(\tau|t + \delta)$ in (23) for $\tau \in [t + \delta, t + T]$. From (16) and (23), $\hat{x}(\tau|t + \delta)$ is described as

$$\begin{aligned} \hat{x}(\tau|t + \delta) &= e^{A_c(\tau-t-\delta)} \hat{x}(t + \delta|t + \delta) \\ &\quad + \int_0^{\tau-t-\delta} e^{A_c s} B \tilde{u}(\tau - s|t + \delta) ds \\ &= e^{A_c(\tau-t-\delta)} \hat{x}(t + \delta|t + \delta) \\ &\quad + \int_0^{\tau-t-\delta} e^{A_c s} B \tilde{u}^*(\tau - s|t) ds \\ &= e^{A_c(\tau-t-\delta)} \hat{x}(t + \delta|t) \\ &\quad + \int_0^{\tau-t-\delta} e^{A_c s} B \tilde{u}^*(\tau - s|t) ds \\ &\quad + e^{A_c(\tau-t-\delta)} [\hat{x}(t + \delta|t + \delta) - \hat{x}(t + \delta|t)] \\ &= \hat{x}(\tau|t) + e^{A_c(\tau-t-\delta)} [x(t + \delta) - \hat{x}(t + \delta|t)]. \end{aligned}$$

Thus, from (16) and (17), we have

$$\begin{aligned} \hat{x}(\tau|t + \delta) &= \hat{x}(\tau|t) + e^{A_c(\tau-t-\delta)} \int_0^\delta e^{A_c s} B_d d(t + \delta - s) ds \\ &= \hat{x}(\tau|t) + \int_{\tau-t-\delta}^{\tau-t} e^{A_c \sigma} B_d d(\tau - \sigma) d\sigma, \quad (\text{A.9}) \end{aligned}$$

where the variable is changed to $\sigma := \tau - t - \delta + s$. Therefore, from $d(\tau - s) \in D(x(\tau - s))$ and Lemma 3,

$$\begin{aligned} &|\hat{x}_i(\tau|t + \delta)| + \hat{\gamma}_i(\tau, w(\cdot|t + \delta)) \\ &\leq |\hat{x}_i(\tau|t)| + \int_{\tau-t-\delta}^{\tau-t} (\|\zeta_{1i}^T(s)\|_1 \|x(\tau - s)\| \\ &\quad + \|\zeta_{2i}^T(s)\|_1) ds + \hat{\gamma}_i(\tau, w(\cdot|t + \delta)) \\ &\leq |\hat{x}_i(\tau|t)| \\ &\quad + \int_{\tau-t-\delta}^{\tau-t} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau - s|t)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \\ &\quad + \hat{\gamma}_i(\tau, w(\cdot|t + \delta)). \quad (\text{A.10}) \end{aligned}$$

From the definition of $\hat{\gamma}_i$ in (13) and Lemma A.1,

$$\begin{aligned} &\hat{\gamma}_i(\tau, w(\cdot|t + \delta)) \\ &= \int_0^{\tau-t-\delta} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau - s|t + \delta)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \\ &\leq \int_0^{\tau-t-\delta} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau - s|t)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \\ &= - \int_{\tau-t-\delta}^{\tau-t} \left(\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau - s|t)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}^T(s)\|_1 \right) ds \\ &\quad + \hat{\gamma}_i(\tau, w(\cdot|t)). \quad (\text{A.11}) \end{aligned}$$

Therefore, from (A.10) and (A.11),

$$\begin{aligned} &|\hat{x}_i(\tau|t + \delta)| + \hat{\gamma}_i(\tau, w(\cdot|t + \delta)) \\ &\leq |\hat{x}_i(\tau|t)| + \hat{\gamma}_i(\tau, w(\cdot|t)). \quad (\text{A.12}) \end{aligned}$$

This implies that, if the solution $\tilde{u}^*(\tau|t)$ satisfies

$$x(\tau|t) \in X(\tau, w(\cdot|t)), \quad \tau \in [t, t + T]$$

at the current time t , then $\tilde{u}(\tau|t + \delta)$ in (23) satisfies

$$\hat{x}(\tau|t + \delta) \in \hat{X}(\tau, w(\cdot|t + \delta)), \quad \tau \in [t + \delta, t + T]$$

at the next time step $t + \delta$. Similarly to (A.12), we have

$$\begin{aligned} &|\tilde{u}_i(\tau|t + \delta) + K_i \hat{x}(\tau|t + \delta)| + \hat{\eta}_i(\tau, w(\cdot|t + \delta)) \\ &\leq |\tilde{u}_i^*(\tau|t) + K_i \hat{x}(\tau|t)| + \hat{\eta}_i(\tau, w(\cdot|t)). \end{aligned}$$

Thus, if $\tilde{u}^*(\tau|t)$ satisfies that

$$\tilde{u}^*(\tau|t) + K \hat{x}(\tau|t) \in \hat{U}(\tau, w(\cdot|t)), \quad \tau \in [t, t + T],$$

then $\tilde{u}(\tau|t + \delta)$ satisfies

$$\begin{aligned} &\tilde{u}(\tau|t + \delta) + K \hat{x}(\tau|t + \delta) \in \hat{U}(\tau, w(\cdot|t + \delta)), \\ &\quad \tau \in [t + \delta, t + T]. \end{aligned}$$

Moreover, it is clear from Lemma A.1 that, if the constraint $w(\tau|t) \leq \omega$ ($\tau \in [t, t + T]$) is satisfied, it is also satisfied that

$$w(\tau|t + \delta) \leq \omega, \quad \tau \in [t + \delta, t + T], \quad (\text{A.13})$$

which concludes the proof for the feasibility at $\tau \in [t + \delta, t + T]$. Note that, similar to (A.13), it follows from $w(t + T|t) \leq 1$ and Lemma A.1 that

$$w(t + T|t + \delta) \leq 1. \quad (\text{A.14})$$

Next, we prove

$$\begin{aligned} &w(\tau|t + \delta) \leq \omega, \\ &w(\tau|t + \delta) \leq 1, \quad \tau \in [t + T, t + T + \delta] \quad (\text{A.15}) \end{aligned}$$

as follows: In the case where $w(t + T|t + \delta) > a_2/a_1$, $w(\tau|t + \delta)$ ($\tau \geq t + T$) is decreasing for $\tilde{u}(\tau|t + \delta) = 0$ ($\tau \geq t + T$) as in (9). Therefore, (A.15) is obviously satisfied from (A.13) and (A.14). On the other hand, in the case where $w(t + T|t + \delta) \leq a_2/a_1$, $w(\tau|t + \delta)$ ($\tau \geq t + T$) cannot be greater than a_2/a_1 for $\tilde{u}(\tau|t + \delta) = 0$ ($\tau \geq t + T$) as in (9). Therefore, (A.15) is proved, since

$$\frac{a_2}{a_1} = \sqrt{\underline{\lambda}(P)} \frac{\alpha_2}{\alpha_1} < 1 \leq \omega$$

from Assumption 1' and 2. Since the conditions in Assumption 1' are written as $1 \leq \underline{\gamma} \sqrt{\underline{\lambda}(P)}$ and $\max_i \|K_i^T\| \leq \underline{\eta} \sqrt{\underline{\lambda}(P)}$ by using $\underline{\gamma}$ and $\underline{\eta}$ in Lemma A.2, we have

$$\begin{aligned} &|x_i| \leq \|x\| \leq \frac{1}{\sqrt{\underline{\lambda}(P)}} \leq \underline{\gamma}, \\ &|K_i x| \leq \|K_i^T\| \|x\| \leq \frac{\|K_i^T\|}{\sqrt{\underline{\lambda}(P)}} \leq \underline{\eta} \quad \forall x \in X_f. \quad (\text{A.16}) \end{aligned}$$

Therefore, it follows Lemma A.2 and (A.16) that

$$\begin{aligned} &|x_i| \leq \gamma_i - \hat{\gamma}_i(\tau, w(\cdot|t + \delta)), \\ &|K_i x| \leq \eta_i - \hat{\eta}_i(\tau, w(\cdot|t + \delta)) \quad \forall x \in X_f. \quad (\text{A.17}) \end{aligned}$$

Since it is clear from (A.15) and Lemma 3 that $\hat{x}(\tau|t + \delta) \in X_f$ ($\tau \in [t + T, t + T + \delta]$), we have

$$\begin{aligned} &\hat{x}(\tau|t + \delta) \in \hat{X}(\tau, w(\cdot|t + \delta)), \\ &\tilde{u}(\tau|t + \delta) = Kx(\tau|t + \delta) \in \hat{U}(\tau, w(\cdot|t + \delta)) \quad (\text{A.18}) \end{aligned}$$

from (A.17) and $\tilde{u}(\tau|t + \delta) = 0(\tau \geq t + T)$. Therefore, (A.15) and (A.18) prove the feasibility for $\tau \in [t + T, t + T + \delta]$, which concludes the proof. \square

Note that (A.17) implies (21) from the definition of \hat{X} , \hat{U} .

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