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Brief Paper

Prediction error methods for limit cycle data ☆

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Abstract

Prediction error methods are considered for identification of the forward linear dynamics of nonlinear feedback closed-loop systems which operate in a perturbed stable limit cycle. A model of the signals measured in a neighborhood of the limit cycle is presented and shown to satisfy a quasistationarity property. Quasistationarity is then used to prove that prediction error methods are both convergent and consistent for our data model. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Closed-loop identification; Limit cycle; Prediction error methods

"Yeah, this is a story of a famous dog,
For the dog that chases its tail will be dizzy."

George Clinton, *Atomic Dog*

1. Introduction

Nonlinear systems operating in a limit cycle are often modeled by an unforced closed-loop system (see Fig. 1) with forward dynamics G and a feedback nonlinearity N (Mees, 1981).[1] For instance, combustion chambers which sustain stable pressure oscillations without external forcing are well modeled by this type of limit cycle system (Murray et al., 1998). In a combustion chamber, pressure oscillations, represented by the signal y translate to heat release u via the nonlinear map N. Heat release is coupled to pressure via a linear acoustic model G. Disturbances to the system are modeled by the noise model H driven by white noise. Identification of the linear forward dynamics for a combustion chamber from heat release and pressure data is a crucial initial step towards designing a feedback controller for suppressing these pressure oscillations. The combustion chamber problem motivates studying identification algorithms for this type of closed-loop system. This work investigates the behavior of the popular prediction error methods (PEM) when used with sampled perturbed limit cycle signals u and y to identify the linear dynamics G and the noise model H. The literature on linear system closed-loop identification is not clear about parametric identification when the unperturbed $(r(t)=0, e(t)=0)$ closed-loop system produces steady, self-excited oscillations. The main work in the area (Ljung, 1978) uses a *sufficient* condition of exponential stability of the closed-loop system to the origin to guarantee convergence of the identification criterion as the amount of data grows. This condition is certainly violated by limit cycle systems which are not globally exponentially stable to the origin. Trajectories of a nonlinear system may have different orbits depending on initial conditions and disturbances, so that the identification criterion, which is based on the spectrum of filtered data, may be different for different experiments. Our intention is to show that PEM are

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[1] Due to the absence of a reference signal ($r(t)=0$ in Fig. 1) designation of the *forward* and *feedback* labels is arbitrary. There exist situations where a reference signal may be present, or N may be a controller, making the definition more sensible.

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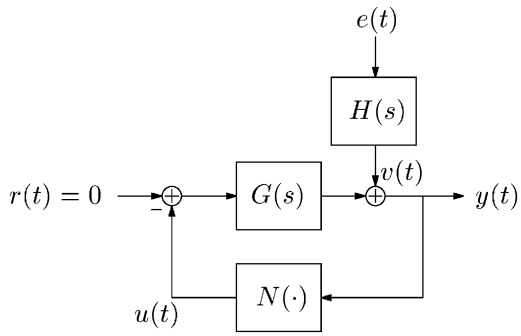

Fig. 1. Identification model structure for PEM.

also valid in the limit cycle case for experiments that always collect data from a neighborhood of an *exponentially asymptotically stable* limit cycle which is perturbed by a small amount of noise. We model the effect of noise on directions parallel to and perpendicular to the limit cycle. By considering data of this particular model it is possible to show convergence of the identification criterion, and consequently feasibility of PEM with limit cycle data. The fundamental property used is that such data satisfy quasistationarity and ergodicity conditions. The study begins with a brief overview of PEM and continues with an investigation of the properties of signals collected around a stable limit cycle. In the presence of a small amount of white noise, disturbances act in different ways along the stable and center manifolds of the asymptotically stable limit cycle resulting in small perturbations of the limit cycle waveforms and a slow, unbiased phase drift. Spectra of signals represented by this model consist of bell-shaped distributions of frequencies centered about the harmonics of average periodic signals. We then show that for this data model PEM estimates of the linear forward system and the noise model are:

- *convergent*, as the available number of data grows, and
- *consistent*, in case the underlying system truly belongs to the identification model class.

2. Background

2.1. Prediction error methods

This section describes PEM for identification of linear discrete time systems. Our objective is to perform discrete time identification of the continuous time systems G and H using sampled data from the continuous time oscillator. Thus, we will use sampled input/output measurements $u_n = u(n\Delta)$ and $y_n = y(n\Delta)$ (for some sampling period $\Delta > 0$ and after appropriate anti-aliasing filtering). Conversion from a discrete time system (identified from data) back to an *approximation* of the continuous time system can be done in several ways (Åström & Wittenmark, 1997). One begins by stipulating the existence of a discrete time linear model

Table 1
Description of operators and functions

Notation	Description
$EX = \int_{\xi\in\mathbb{R}^n} \xi f_X(\xi)\,\mathrm{d}\xi$	Expectation operator
$\Lambda x_n = \lim_{L\to\infty} \frac{1}{L}\sum_{n=1}^{L} x_n$	Averaging operator
$R_s(\ell) = \Lambda R_s(n, n-\ell)$	Autocorrelation of s_n
$R_s(n,m) = E s_n s_m^T$	
$\Phi_s(v) = \sum_{\ell=-\infty}^{\infty} R_s(\ell)\mathrm{e}^{-jv\ell}$	Spectrum of s_n
$\Phi_{sw}(v) = \sum_{\ell=-\infty}^{\infty} R_{sw}(\ell)\mathrm{e}^{-jv\ell}$	Cross-spectrum of s_n, w_n
$R_s(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(v)\,\mathrm{d}v$	Parseval's relation

structure

$$y_n = G_\theta(z)u_n + H_\theta(z)e_n, \quad \theta \in D, \tag{1}$$

where the θ dependence indicates that G_θ and H_θ belong to some parametrized family of rational transfer functions (over a region of parameters D). H_θ is assumed to be stable monic (that is, $H_\theta(\infty) = 1$) and to have a stable inverse. We will also only deal with forward linear systems G_θ that have a delay, a condition which guarantees well-posedness of the closed-loop system. The perturbation $e_n = e(n\Delta)$ is white noise. Based on this model one defines a one-step-ahead prediction of future values of the output from input and past output data

$$\hat{y}_n(\theta) = H_\theta^{-1}(z)G_\theta(z)u_n + (1 - H_\theta^{-1}(z))y_n$$

as well as the prediction error $\varepsilon_n(\theta) = y_n - \hat{y}_n$. Note that if the output y_n is truly described by (1) with a known θ then $\varepsilon_n = e_n$ after dissipation of initial conditions. Given a finite number of data, say (u_n, y_n) for $n = 1, 2, \ldots, L$, it is possible to obtain a parameter estimate

$$\theta_L^\star = \arg\min_{\theta\in D} V_L(\theta),$$

where $V_L(\theta) = \frac{1}{L}\sum_{n=1}^{L} \varepsilon_n^2(\theta)$. For simplicity we have assumed single-input single-output, real-valued systems, so $\varepsilon_n(\theta)$ is a real-valued scalar.

2.2. Quasistationarity

We follow the identification framework in Ljung (1999). The main signal class of interest is the class of quasistationary signals defined below. Refer to Table 1 for definitions of useful operators and functions.

Definition 1 (Quasistationarity). A sequence of column vectors $\{s_n\}_{n\in\mathbb{Z}}$ is said to be quasistationary if

(i) $Es_n = \bar{s}_n$, $|\bar{s}_n| < C_1 < \infty$ for all $n \in \mathbb{Z}$, and
(ii) $|R_s(n,m)| < C_2 < \infty$, and $R_s(\ell)$ exists for all $\ell \in \mathbb{Z}$ independent of n.

2.3. Properties of perturbed limit cycle data

"*... fluctuations in directions with different eigenvalues will affect the dynamics differently...Fluctuations parallel to eigenvectors with negative eigenvalues will equilibrate to a metastable state in which the contraction is balanced by the noise-induced expansion...Fluctuations along neutrally stable eigenvectors are similar to an unbiased random walk, and will result in diffusion*." (Knobloch & Weiss, 1989).

This section provides a qualitative picture of the dynamics of trajectories of a nonlinear system disturbed by noise, in the neighborhood of a stable limit cycle. This picture results in a useful model of the signals encountered in the closed-loop system. Consider a uniformly (in γ) exponentially, asymptotically stable limit cycle governed by the m-dimensional vector differential equation

$$\dot{\gamma}(t) = f(\gamma(t)).$$

We will investigate the behavior of the system in Fig. 1 expressed in this form. Our model assumes a strictly proper rational transfer function $G(s)$ and a static memoryless nonlinearity $N \in C^m$, yielding $f \in C^1$.

For a given point $\gamma(t)$ in the neighborhood of the limit cycle, define the coordinate system $(\gamma^{\|}(t), \gamma^{\perp}(t))$ as the nearest orthogonal projection from $\gamma(t)$ to the limit cycle (i.e. $\gamma^{\|}(t)$) and its orthogonal difference (i.e. $\gamma^{\perp}(t)$). Refer to Fig. 2. The Center Manifold Theorem for Periodic Orbits (Perko, 1991) states that this is a legitimate orthogonal coordinate system. These coordinates are a point on the center manifold defined by the limit cycle and a point in the stable manifold of the limit cycle. We now consider the dynamics of the disturbed system

$$\begin{pmatrix} \dot{\gamma}^{\|}(t) \\ \dot{\gamma}^{\perp}(t) \end{pmatrix} = g(\gamma^{\|}(t), \gamma^{\perp}(t)) + d(t)$$

in this new coordinate system, where $d(t)$ captures the effect of external disturbances to the system. Examples of similar decompositions for analyzing the effect of noise in nonlinear *planar* systems can be found in Daffertshofer (1998) and Kurrer and Schulten (1991). We use Taylor's Theorem to expand the right-hand side about the point $(\gamma^{\|}(t), 0)$ on the limit cycle at a given time associated with $\gamma(t)$

$$\begin{aligned} \begin{pmatrix} \dot{\gamma}^{\|} \\ \dot{\gamma}^{\perp} \end{pmatrix} &= g(\gamma^{\|}, 0) + \left.\frac{\partial g}{\partial \gamma^{\|}}\right|_{(\gamma^{\|},0)} \cdot 0 + \left.\frac{\partial g}{\partial \gamma^{\perp}}\right|_{(\gamma^{\|},0)} \cdot \gamma^{\perp} \\ &\quad + o(|\gamma^{\perp}|) + d \\ &= \begin{pmatrix} U(\gamma^{\|}) \\ 0 \end{pmatrix} + \begin{pmatrix} V(\gamma^{\|}) \\ W(\gamma^{\|}) \end{pmatrix} \gamma^{\perp} + o(|\gamma^{\perp}|) + \begin{pmatrix} d^{\|} \\ d^{\perp} \end{pmatrix} \end{aligned}$$

(omitting momentarily the time t dependence). Note that $\dot{\gamma}^{\|}(t) = U(\gamma^{\|}(t))$ with $\gamma^{\perp}(t) = 0$ defines the unperturbed limit cycle. Let $\gamma_n^{\|} = \gamma^{\|}(n\mu)$, $\gamma_n^{\perp} = \gamma^{\perp}(n\mu)$, etc., for a small stepsize $0 < \mu \ll 1$. The limit cycle *phase coordinate* is

$$\begin{aligned} \gamma_{n+1}^{\|} &= \gamma_n^{\|} + \mu U(\gamma_n^{\|}) + \mu V(\gamma_n^{\|})\gamma_n^{\perp} + o(|\mu\gamma_n^{\perp}|) \\ &\quad + \mu d_n^{\|} + o(\mu). \end{aligned}$$

The coordinate $\gamma_n^{\|}$ is the position of the perturbed limit cycle on the center manifold. The unperturbed system governing this trajectory has unity eigenvalue, so that the trajectory perturbed by $d_n, \gamma_n^{\perp}$ and higher-order terms is well described by a *random walk*. For the motion orthogonal to the phase $\gamma^{\|}$ we write

$$\gamma_{n+1}^{\perp} = \gamma_n^{\perp} + \underbrace{\mu W(\gamma_n^{\|})\gamma_n^{\perp}}_{\triangleq \alpha(n, \gamma_n^{\perp})} + \underbrace{o(|\mu\gamma_n^{\perp}|) + \mu d_n^{\perp} + o(\mu)}_{\triangleq \beta(n, \gamma_n^{\perp})}.$$

If α and β as defined here satisfy the conditions of Theorem 5.1 in LaScala, Bitmead, and James (1995) (where the notation uses $\alpha \mapsto f(\cdot)$ and $\beta \mapsto g(\cdot)$) the motion $\gamma_n^{\perp}$ remains bounded, and thus trajectories of the overall system live in a tube surrounding the center manifold. In summary: we have made strong assumptions on the stability of the system on the stable manifold, and required bounded higher-order terms in a neighborhood of the limit cycle as well as bounded noise. Our derivations so far focus on the separate evolution of the center manifold component $\gamma^{\|}$ and the stable manifold components $\gamma^{\perp}$. This relies on considering the noise terms $d^{\|}$ and $d^{\perp}$ as dominant and independent drivers for the perturbed systems and, additionally, that the motion along the limit cycle is smooth. To proceed further, in the next section we formalize this behavior by assuming a model for the signals collected from the perturbed limit cycle system. The conclusions flowing from these assumptions will shortly be tested on an example.

2.4. Sampled perturbed limit cycle data

The analysis of trajectories of the system in the coordinate system defined by center and stable manifolds shows that data collected from perturbed orbits (i.e. u_n, y_n) is well described by the model

$$x_n = \bar{x}(n\Delta + \phi + \varphi_n) + \tilde{x}_n, \tag{2}$$

where $\bar{x}(t)$ is a periodic signal (period T, frequency $\omega = 2\pi/T$) of the undisturbed system, $\phi \in [0, T]$ is some fixed phase offset, phase φ_n is a random walk (resulting from perturbations along the center manifold $\gamma^{\|}$), and $\tilde{x}_n$ is a (bounded) disturbance in amplitude (resulting from perturbations along the stable manifold $\gamma^{\perp}$ caused by stable filtering of the noise).

The model in Eq. (2) facilitates spectral analysis of sampled, perturbed limit cycle data. Define $\omega_k = k\omega\Delta$, $|\omega_k| < \pi$ for $|k| \leqslant K$ and assume that harmonics with frequency higher than $1/2\Delta$ are negligible. This leads to the assumptions on the data collected from the loop that follow.

(A0) Signals sampled from the system in Fig. 1 have the form

$$x_n = \sum_{|k| \leqslant K} \chi_k e^{j\omega_k(n+\phi+\varphi_n)} + \tilde{x}_n, \tag{3}$$

(A1) where phase perturbations φ_n are given by a Brownian motion $\varphi_n = \sum_{i=0}^{n} w_i$ with $w_n \sim N(0, \sigma_w^2)$, w_n i.i.d..

(A2) Amplitude perturbations $\tilde{x}_n$ are a result of stable filtering of white noise $e_n = e(n\Delta)$, $|e_n| < C$ for all $n \in \mathbb{Z}$, and are thus quasistationary with autocorrelation function $R_{\tilde{x}}(\ell)$ and spectrum $\Phi_{\tilde{x}}(v)$.

(A3) Phase offset ϕ, phase perturbations φ_n and amplitude perturbations $\tilde{x}_m$ are independent and thus uncorrelated, i.e.

$$E e^{j\omega_k(n+\phi+\varphi_n)} \tilde{x}_m = 0 \quad \forall n, m \in \mathbb{Z}.$$

The theorem that follows states that signals described by the model above are quasistationary, thus accommodating the PEM framework.

Theorem 1. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence of vectors satisfying* (A0)–(A3). *Then, $\{x_n\}_{n\in\mathbb{Z}}$ is quasistationary and has autocorrelation and spectrum*

$$R_x(\ell) = \sum_{|k| \leqslant K} \chi_k \chi_k^H e^{j\omega_k \ell} D(k, \ell) + R_{\tilde{x}}(\ell),$$

$$\Phi_x(v) = \sum_{|k| \leqslant K} \chi_k \chi_k^H \Phi_D(k, v - \omega_k) + \Phi_{\tilde{x}}(v),$$

where

$$D(k, \ell) \triangleq e^{-(|\ell|/2)\omega_k^2 \sigma_w^2}, \quad \Phi_D(k, v) = \sum_{\ell=-\infty}^{\infty} D(k, \ell) e^{-jv\ell}.$$

Proof. See Appendix A. □

The theorem shows that the spectra of the perturbed limit cycle data consists of bell-shaped distributions of frequencies centered about the harmonics of average periodic signals. Thus, perturbations *blur* the line spectrum of the unperturbed limit cycle system. Another important assumption to be adopted deals with the *ergodic* behavior of quasistationary signals of the form (3) from assumption (A0) and their stably filtered versions. It will often be the case that identification based on the criterion $V_L(\theta)$ is conducted from a single sequence of input/output pairs $\{(u_n, y_n)\}_{1 \leqslant n \leqslant L}$. It is natural to expect two properties from the identification procedure: (i) if the amount of data grows the identification criterion converges to a well-defined limit over all possible parametrizations $\theta \in D$ and (ii) if the experiment is repeated the criterion converges to the same limit. Since $V_L(\theta)$ is based on statistics (moments, correlations, spectra, etc.) we are asking for the statistics gathered from a single realization (or experiment) to converge as $L \to \infty$ to the statistics computed over an ensemble of realizations. To satisfy this requirement we make the ergodic assumption:

(E) Time averages of the moments (first and second moments) of quasistationary signals s_n, w_n of type (3) converge to ensemble averages

$$\Lambda s_n = \Lambda(E s_n), \quad \Lambda s_n w_m^T = \Lambda(E s_n w_m^T).$$

Expectations are taken over the random disturbances. This assumption indirectly requires that disturbances provide a *mixing property* whereby events which are separated by long amounts of time are independent. Using this property, we conclude this section with a convergence theorem.

Theorem 2. *Let $\{G_\theta(z)\}_{\theta \in D}$ be a family of stable rational filters $G_\theta(z) = \sum_{i=0}^{\infty} g_i(\theta) z^{-i}$. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence of vectors satisfying* (A0)–(A3) *and* (E). *Then each sequence $\{s_n(\theta)\}_{n\in\mathbb{Z}}$, where $s_n(\theta) = G_\theta(z) x_n$, is quasistationary and has a well-defined autocorrelation function $R_s(\ell;\theta)$ and spectrum*

$$\Phi_s(v;\theta) = G_\theta(e^{jv}) \Phi_x(v) G_\theta^T(e^{-jv}).$$

Furthermore, as $L \to \infty$,

$$\sup_{\theta \in D} \left\| \frac{1}{L} \sum_{n=1}^{L} s_n(\theta) s_n^T(\theta) - R_s(\ell;\theta) \right\|_F \to 0.$$

Proof. The first statement of the theorem is Theorem 2.2 of Ljung (1999), proven on pp. 52–54, and the second statement is a corollary proven on pp. 54. □

3. Statistical properties of PEM

3.1. Convergence

By Theorems 1 and 2, the prediction error expressed as the output of a stable linear system $\varepsilon_n(\theta) = T_\theta(z) x_n$ is quasistationary, where $x_n = (u_n, y_n)^T$ and u_n, y_n satisfy assumptions (A0)–(A3) and (E), and $T_\theta(z) = (-H_\theta^{-1}(z) G_\theta(z), H_\theta^{-1}(z))$. By Theorem 2 we can say that the limit

$$V(\theta) \triangleq \lim_{L\to\infty} V_L(\theta) = R_\varepsilon(0;\theta)$$

exists uniformly in $\theta \in D$ and that $\theta_L^\star \to \arg\min_{\theta\in D} V(\theta)$ as $L \to \infty$ for compact D.

3.2. Consistency

The spectrum of the prediction error is given by Forsell (1999) (SISO case)

$$\Phi_\varepsilon(v,\theta) = \frac{1}{|H_\theta|^2} \begin{pmatrix} G_0 - G_\theta \\ H_0 - H_\theta \end{pmatrix}^T \underbrace{\begin{pmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Phi_e \end{pmatrix}}_{\Phi} \begin{pmatrix} G_0 - G_\theta \\ H_0 - H_\theta \end{pmatrix}^*$$

(the frequency v dependence is omitted) so that the average prediction error is $V(\theta) = \pi \int_{-\pi}^{\pi} \Phi_\varepsilon(v, \theta)\,dv$. Note that white perturbations $e(t)$ have $\Phi_e(v) = \lambda \geqslant 0$. As long as $\Phi(v) > 0$ for all $v \in [-\pi, \pi]$ the system can be recovered, i.e. $G_\theta = G_0$ and $H_\theta = H_0$. A requirement for consistency is that the data must carry information about all modes of a particular system. In closed loop, insufficient information in the data for identification of the parameters of a particular model may happen as a result of a linear time invariant relationship between u and y (Gustavsson, Ljung & Söderström, 1977). In this case, we say that the experiment is not *informative* enough with respect to the model structure. Two ways of getting around this problem are to use a persistently exciting reference signal (i.e. $r(t)$ in Fig. 1) or time varying feedback designed to make the experiment informative. By alternating between different linear controllers (two controllers for the single-input single-output case) it is possible to design an experiment which provides sufficient information for identification (Ljung, 1999). In our setting the facility for manipulating the reference signal or the feedback is not available. On the other hand, the system of interest has nonlinear feedback, which "...should, in general, yield experiments that are informative enough" (Ljung, 1999). Our analysis shows that positive definiteness of the data spectrum matrix for all frequencies serves as a sufficient condition for consistency. Thus, we count on the feedback nonlinearity to provide the richness needed for identifiability if the harmonic content of the limit cycle is sufficient.

4. Example

Consider the system of Fig. 1 with components

$$G(s) = \frac{4(s+20)^2}{(s+1)(s+2)(s+3)}, \quad H(s) = \frac{s+30}{s+0.7}$$

and $N(x) = (2/\pi)\tan^{-1}(x)$. We simulate in discrete time using a sample period $\Delta = 0.01$ and zero-order-hold (ZOH) versions of the transfer functions

$$G_0(z) = \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}},$$

$$H_0(z) = \frac{1 + c_1 z^{-1}}{1 + d_1 z^{-1}}$$

with parameters $b_1 = 0.0469$, $b_2 = -0.0768$, $b_3 = 0.0314$, $a_1 = -2.9407$, $a_2 = 2.8825$, $a_3 = -0.9418$, $c_1 = -0.7010$, and $d_1 = -0.9930$. This ensures that the identification model structure exactly describes the plant and noise model.

We conduct experiments in two scenarios: in the absence of noise and with uniformly distributed white noise $e_n \sim U[-0.02, 0.02]$. Phase space trajectories are shown in Fig. 3. Fig. 3 indicates one way that the analysis might go wrong—the noise might be so large as to drop you discontinuously onto an entirely different part of the limit cycle which cannot be explained by random walk phase shifts. Spectra of the noisy and noiseless outputs y are shown in Fig. 4. The perturbed output spectrum consists of bell-shaped distributions centered around the harmonics of the noiseless output as predicted in the analysis. The noiseless experiment results in exact recovery of G_0. Fig. 5 shows the percentage parameter errors for multiple

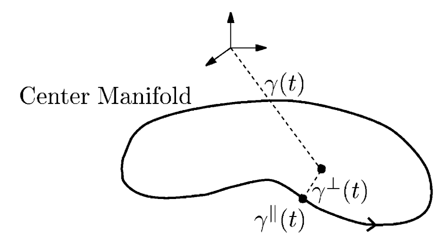

Fig. 2. Co-ordinate system based on center manifold.

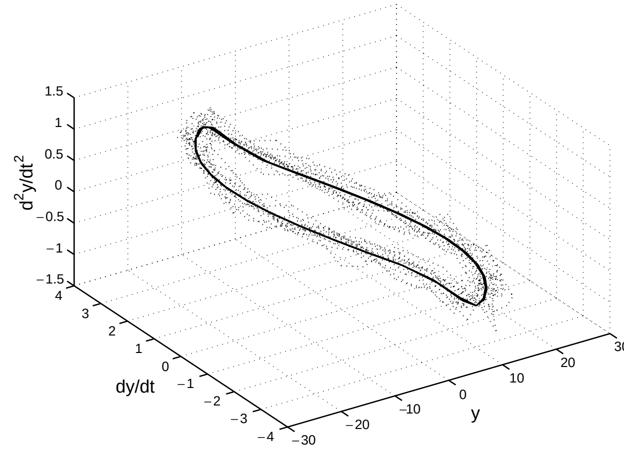

Fig. 3. Phase space: noiseless (solid) and perturbed (dotted) limit cycle data.

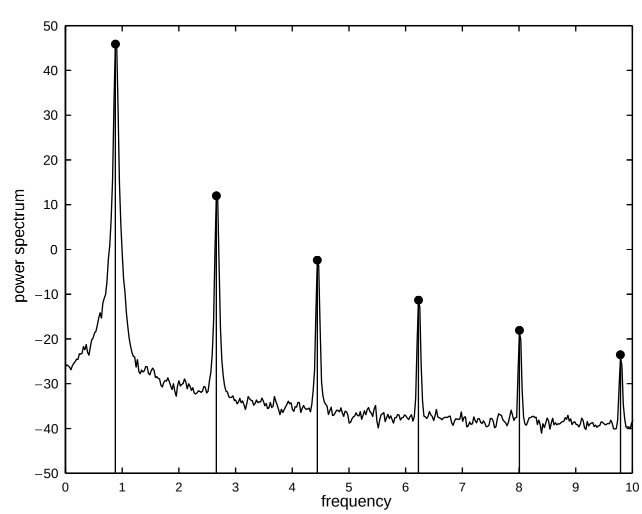

Fig. 4. Power spectrum of system output for noiseless case (stems), and perturbed case (solid line).

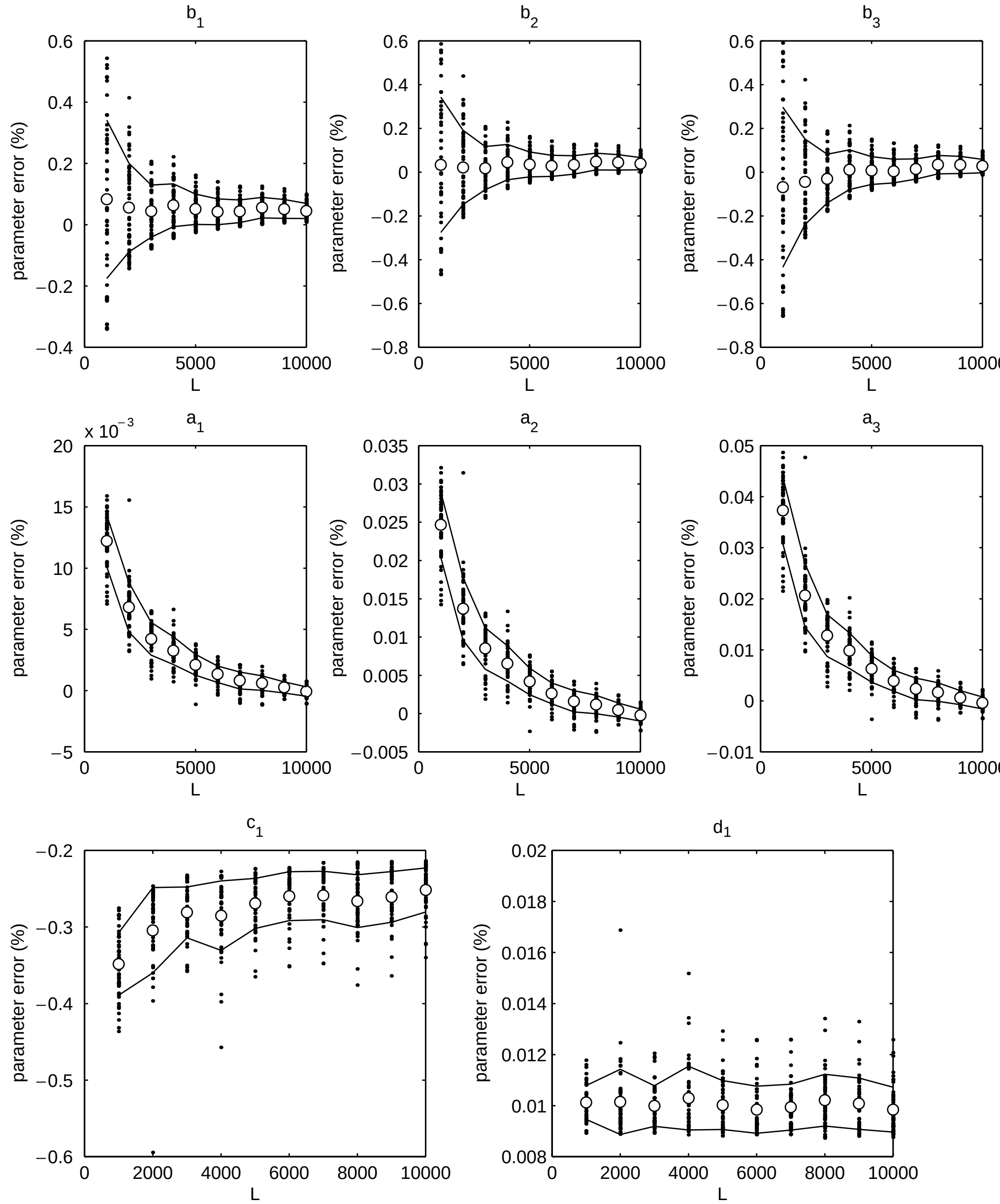

Fig. 5. (·) Parameter error percentages for 100 experiments performed at various data record lengths. (∘) Average parameter error percentages. (−) Averages plus and minus one standard deviation.

noisy experiments performed with increasing data record lengths. The monotonic decay of the parameter errors indicates that PEM are indeed appropriate for noisy scenarios, as expected from the obtained consistency results.

5. Conclusion

We have seen that data collected from a perturbed limit cycling nonlinear feedback system can be valid for use with PEM. In the absence of output disturbances, the data is periodic and therefore quasistationary. Data from a limit cycle system excited by a small amount of white noise can be well modeled as quasistationary by analyzing the effects of perturbation along the stable and center manifolds of the limit cycle. By way of similar analysis, it may be possible to show that data collected from other types of (perturbed or unperturbed) nonlinear attractors, such as chaotic attractors, also fit the PEM framework. A general theory that demonstrates the validity of PEM for these complex nonlinear dynamical systems would be valuable in the area of nonlinear system identification, which is still in its infancy.

Appendix A

Proof (*of Theorem 1*). We begin with the autocorrelation function of a signal x_n of type (3)

$$\begin{aligned} R_x(n,m) &= E x_n x_m^T \\ &= E(\bar{x}(n+\phi+\varphi_n)+\tilde{x}_n)(\bar{x}(m+\phi+\varphi_m)+\tilde{x}_m)^T \\ &= E\bar{x}(n+\phi+\varphi_n)\bar{x}(m+\phi+\varphi_m)^T \\ &\quad + \underbrace{E\bar{x}(n+\phi+\varphi_n)\tilde{x}_m^T + E\tilde{x}_n\bar{x}(m+\phi+\varphi_m)^T}_{0 \text{ by (A3)}} \\ &\quad + \underbrace{E\tilde{x}_n\tilde{x}_m^T}_{\triangleq R_{\tilde{x}}(n,m)}. \end{aligned}$$

Write

$$\begin{aligned} & E\bar{x}(n+\phi+\varphi_n)\bar{x}(m+\phi+\varphi_m)^T \\ &= E\left(\sum_{|k|\leqslant K}\chi_k e^{j\omega_k(n+\phi+\varphi_n)}\right)\left(\sum_{|k'|\leqslant K}\chi_{k'} e^{j\omega_{k'}(m+\phi+\varphi_m)}\right)^T \\ &= \sum_{\substack{|k|\leqslant K\\|k'|\leqslant K}}\chi_k\chi_{k'}^T e^{j(\omega_k n+\omega_{k'}m)} e^{j(\omega_k+\omega_{k'})\phi} E e^{j(\omega_k\varphi_n+\omega_{k'}\varphi_m)}. \end{aligned}$$

Assume w.l.o.g. $n \geqslant m$. By assumption (A1), when $n > m$

$$\varphi_m = \underbrace{\sum_{i=0}^{m} w_i}_{\triangleq x}, \quad \varphi_n = \underbrace{\sum_{i=0}^{m} w_i}_{x} + \underbrace{\sum_{i=m+1}^{n} w_i}_{\triangleq y}$$

so that x and y are independent Gaussian random variables with $x \sim N(0,(m+1)\sigma_w^2)$ and $y \sim N(0,(n-m)\sigma_w^2)$. If $n=m$ then $y=0$. We can simplify the expression $E(\cdot)$ using independence of x and y which results in

$$E e^{j(\omega_k x+\omega_{k'}(x+y))} = E e^{j(\omega_k+\omega_{k'})x} E e^{j\omega_{k'}y}.$$

The two expectations may be computed by interpreting them as the characteristic functions of Gaussian random variables (Leon-Garcia, 1989), i.e. if $Z \sim N(0,\sigma^2)$ then $E e^{j\omega z} = e^{-\omega^2\sigma^2/2}$ and therefore

$$E e^{j(\omega_k\varphi_n+\omega_{k'}\varphi_m)} = e^{-[(m+1)/2)]\sigma_w^2(\omega_k+\omega_{k'})^2} e^{-[(n-m)/2]\sigma_w^2\omega_{k'}^2}.$$

We are ready to compute the autocorrelation function. Begin with $\ell \geqslant 0$ so

$$\begin{aligned} R_x(\ell) &= \Lambda R_x(n,n-\ell) \\ &= \lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\sum_{\substack{|k|\leqslant K\\|k'|\leqslant K}}\chi_k\chi_{k'}^T e^{j(\omega_k n+\omega_{k'}(n-\ell))} e^{j(\omega_k+\omega_{k'})\phi} \\ &\qquad E e^{j(\omega_k\varphi_n+\omega_{k'}\varphi_{n-\ell})} + \Lambda R_{\tilde{x}}(n,n-\ell) \\ &= \sum_{\substack{|k|\leqslant K\\|k'|\leqslant K}}\chi_k\chi_{k'}^T e^{j(\omega_k+\omega_{k'})\phi} e^{-j\omega_{k'}\ell} e^{-\ell/2\sigma_w^2\omega_{k'}^2} \\ &\quad \times\left(\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L} e^{j(\omega_k+\omega_{k'})n} e^{-[(n-\ell+1)/2)]\sigma_w^2(\omega_k+\omega_{k'})^2}\right) \\ &\quad + R_{\tilde{x}}(\ell). \end{aligned}$$

Note that $R_{\tilde{x}}(\ell)$ is well defined since it consists of filtering of white noise through a stable system, by assumption (A2). If $\omega_k \neq -\omega_k'$, or equivalently $k \neq -k'$ we find that

$$\begin{aligned} & \sum_{n=1}^{L} e^{j(\omega_k+\omega_{k'})n} e^{-[(n-\ell+1)/2)]\sigma_w^2(\omega_k+\omega_{k'})^2} \\ &\quad \leqslant \sum_{n=1}^{\infty} |e^{j(\omega_k+\omega_{k'})n}| e^{-[(n-\ell+1)/2)]\sigma_w^2(\omega_k+\omega_{k'})^2} \\ &\quad = K_1\sum_{n=1}^{\infty} e^{-K_2 n} < \infty. \end{aligned}$$

Thus, the limit above converges to zero as L grows for $k \neq -k'$, and to unity when $k=-k'$, yielding

$$R_x(\ell) = \sum_{|k|\leqslant K}\chi_k\chi_k^H e^{j\omega_k\ell} D(k,\ell) + R_{\tilde{x}}(\ell),$$

where $D(k,\ell) \triangleq e^{-(\ell/2)\omega_k^2\sigma_w^2}$. For $\ell < 0$ we arrive at the same expression for $R_x(\ell)$ but with $D(k,\ell) \triangleq e^{\ell/2\omega_k^2\sigma_w^2}$. Thus, for any $\ell \in \mathbb{Z}$ we use $D(k,\ell) \triangleq e^{-(|\ell|/2)\omega_k^2\sigma_w^2}$. We can express the spectrum by noticing that the product of $D(k,\ell)$ with $e^{j\omega_k\ell}$ translates to a convolution of the transform

$$\Phi_D(k,v) = \sum_{\ell=-\infty}^{\infty} D(k,\ell) e^{-jv\ell}$$

and a spectral line at $v=\omega_k$, which simply shifts the center of Φ_D from the origin to ω_k. In summary

$$\Phi_x(v) = \sum_{|k|\leqslant K}\chi_k\chi_k^H\Phi_D(k,v-\omega_k) + \Phi_{\tilde{x}}(v). \qquad \square$$

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