State Space Modeling for MIMO Wireless Channels

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***Abstract*— State space models are proposed to represent MIMO frequency-selective wireless channels with the motivation of better model approximation performance and more parsimonious parametrization. We study the MIMO channel approximation error as a function of model order and show that state-space models possess improved performance compared to more standard FIR models. A recursive algorithm, based on the subspace system identification methods, to estimate the state-space channel model using training data is presented. When compared to the FIR-based Recursive Least Squares algorithm, the state-space-based channel estimator shows the ability of providing low-order models of high-quality channel approximation, while preserving comparable convergence rate.**

I. Introduction

Multi-input multi-output (MIMO) communication systems have been shown to possess the ability of providing significant capacity gain at no cost of extra spectrum in a rich scattering environment [1]. Most capacity-improving space-time processing schemes assume the availability of a channel model at the receiver.

Since any channel model is essentially an approximation of the physical channel, the goal of channel modeling is to find a model that is as close to the real channel as possible and maintains manageable complexity as well. FIR models have been widely used for wireless channels for their simplicity and guaranteed stability. On the other hand, state-space models are able to provide better model approximation performance since they are a more general class of models that includes FIR models as a subset.

Furthermore, when the MIMO channel is frequency-selective, an FIR model can be very non-parsimonious and contains excessive redundancy since it represents the subchannels of a MIMO channel with *separately* parametrized finite-length impulse responses. Thus, in situations where there exists spatial correlation between the fading of the subchannels [2], it may be beneficial to adopt a state-space model which represents the whole channel as a single entity and hence captures the structure in the MIMO channel to allow more parsimonious description of it.

It is the subject of this paper to explore the application of state-space models to represent MIMO frequency-selective wireless channels in the hope for better model approximation performance and more parsimonious parametrization. We quantify the model approximation error using the upper bound of the minimum H_∞ norm of the difference between the original channel and the approximated channel model. It is shown that state-space models always maintain lower H_∞ approximation error than the FIR models with the same model order for either spatially uncorrelated MIMO channels or correlated ones. Particularly, the performance advantage of state-space models becomes more evident as the spatial correlation increases due to their ability to capture the spatial structure.

One concern in using state-space models to represent the channel is that they may result in slower convergence rate of the adaptive channel estimator since the number of parameters to be estimated in a state-space model is generally greater than that of an FIR model of the same order. In this paper, we present a training-based recursive algorithm, originally proposed in [3], for estimating the state-space channel models. Numerical simulation results show that this state-space channel estimation algorithm exhibits comparable convergence rate to the FIR-based Recursive Least Squares channel estimator, and provides low-order models of high-quality channel approximation. The algorithm evolves from the recursive Ordinary MOESP algorithm [4], [5], which belongs to a family of methods called subspace system identification, but differs from it by making modifications necessary for utilizing non-contiguous training data and applying a new RLS-like exponential forgetting scheme on the training data to improve the adaptation rate.

This paper is organized as follows. Section II investigates the channel approximation error of state-space models and compares it to that of FIR models. Special attention is given to spatially correlated channels whose spatial structure is accessible to state-space models, but generally not to FIR models, to produce low-order models of good approximation quality. Section III briefly describes the recursive state-space channel estimation algorithm and provides numerical simulation results that compare its convergence performance to that of the FIR-based RLS channel estimator. The paper is concluded in Section IV.

II. State-space models for MIMO channels

An m-input p-output length-L ISI channel can be well described by a symbol-sampled discrete-time FIR model as follows

$$\mathbf{y}_k = \sum_{i=0}^{L-1} \mathbf{H}_i \mathbf{u}_{k-i} + \mathbf{n}_k, \tag{1}$$

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where
$\mathbf{u}_k$ – $m \times 1$ input vector at time k;
$\mathbf{y}_k$ – $p \times 1$ output vector at time k;
$\mathbf{n}_k$ – $p \times 1$ white Gaussian noise vector at time k;
$\{\mathbf{H}_\tau\}_{\tau=0}^{L-1}$ – matrix channel taps of dimension $p \times m$.

Another approach to representing ISI channels is to use a state-space model

$$\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k \\ \mathbf{y}_k &= C\mathbf{x}_k + D\mathbf{u}_k + \mathbf{n}_k, \end{aligned} \tag{2}$$

where $\mathbf{x}_k$ is the $n \times 1$ state vector at time k.

For the purpose of investigating the channel approximation performance of both state-space and FIR models, a reference channel is generated as the true channel. Then we compare the upper bounds of the minimum H_∞ channel approximation error of both state-space models and FIR models. Numerical results show the superiority of state-space models in providing high-quality channel representation.

A. Generation of Reference Channels

In order to evaluate the channel approximation performance of state-space and FIR models, a true or reference physical channel ought to be given as the reference of comparison. The ideal reference channel would be obtained directly through channel measurement. However, for simulation study, we could also use synthesized symbol-rate-sampled FIR MIMO channel models as the reference.

It may seem that using FIR reference channels is not fair to state-space models since FIR models would have the advantage of possessing the same structure as the reference channel. However, since we never know the true order of the physical channel in practice, any channel model we adopt is only an approximation of the physical channel and generally has a lower order than the true one. Therefore, we can maintain the fairness of the comparison by considering only the approximation performance of state-space or FIR models with orders strictly less than the true order of the reference channel.

The reference channels in this paper are generated using the so-called "correlation channel model", which describes the second-order statistics of the MIMO fading channel by separate transmit antenna covariance matrix and receive antenna covariance matrix. Specifically, the matrix channel taps, $\{\mathbf{H}_\tau\}_{\tau=0}^{L-1}$, can be modeled as [6],

$$\mathbf{H}_\tau = (R_{RX})^{1/2}\mathbf{H}_\mathbf{w}(R_{TX})^{T/2}, \tag{3}$$

where $\mathbf{H}_\mathbf{w}$ is a $p \times m$ matrix with i.i.d. zero-mean circularly-symmetric complex gaussian elements. R_{TX} and R_{RX} are, respectively, the transmit covariance matrix and receive covariance matrix. Under the assumption of uniformly distributed angle of arrival, R_{TX} and R_{RX} can be modeled with closed-form expressions as functions of angular spread Δ, angle of arrival ϕ and normalized antenna spacing D/λ_c, where D is the antenna spacing and λ_c the carrier wavelength. See [7] for detailed development of the closed-form expressions.

	TX/MU	RX/BS
Angular Spread Δ	$180°$	$24°$
Angle of Arrival ϕ	$45°$	$45°$
Baseline Length L_{bs}	0.3 m	1.0 m
Normalized Antenna Spacing D/λ_c	$\frac{L_{bs}}{m\lambda_c}$	$\frac{L_{bs}}{p\lambda_c}$

TABLE I
CONFIGURATION OF THE UNIFORM LINEAR ARRAYS EXPLOITED BY THE BASE STATION (BS) AND THE MOBILE UNIT (MU) IN AN UPLINK TRANSMISSION.

The antenna configuration in this paper, summarized by Table I, is selected to simulate the uplink transmission. We assume that both the base station and the mobile unit exploit uniform linear arrays with fixed baseline length L_{bs} of 1.0 meter and 0.3 meters, respectively. Thus, as the number of antenna elements increase, their spacing would decrease proportionally, which in turn leads to an increased antenna correlation. The length of all the subchannels L is 6.

B. Upper bounds of minimum H_∞ approximation error

A multi-input multi-output linear system can be represented by its transfer function matrix $\mathbf{G}(e^{j\omega})$. The H_∞ norm of any stable system is defined as

$$\|\mathbf{G}\|_\infty \triangleq \max_\omega \bar{\sigma}[\mathbf{G}(e^{j\omega})],$$

where $\bar{\sigma}[\cdot]$ represents the largest singular value of a matrix.

Supposing that $\mathbf{G}(e^{j\omega})$ is the reference channel and $\hat{\mathbf{G}}(e^{j\omega})$ is an approximation of $\mathbf{G}(e^{j\omega})$, then the H_∞ approximation error is given by

$$\epsilon_\infty = \max_\omega \bar{\sigma}[\hat{\mathbf{G}}(e^{j\omega}) - \mathbf{G}(e^{j\omega})]. \tag{4}$$

The order of a MIMO system, as indicated by the McMillan degree of its transfer function matrix, is equal to the number of *Hankel singular values* of the system. Assuming that the Hankel singular values of an nth-order system are given by

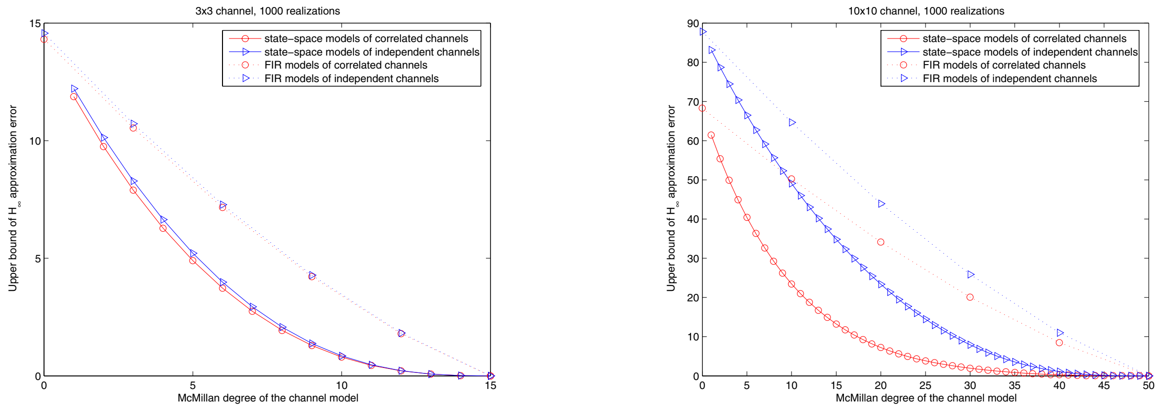
$$\sigma_1(\mathbf{G}) \geq \sigma_2(\mathbf{G}) \geq \cdots \geq \sigma_n(\mathbf{G}),$$

a common approach to produce a pth-order ($p < n$) approximation of the original system is to keep only the largest p Hankel singular values $\{\sigma_i(\mathbf{G})\}_{i=1}^p$ and remove the relatively small ones. As shown by [8] Theorem 7.11, the H_∞ norm of the difference between a MIMO system and its optimal low-order state-space approximation is upperbounded by the sum of the neglected Hankel singular values,

$$\epsilon_{\infty,opt} \leq \sum_{i=p+1}^{n} \sigma_i(\mathbf{G}). \tag{5}$$

See [8] for a detailed treatment of model approximation and Hankel singular values.

However, if we exert the constraint that the low-order approximated model must have finite-length impulse responses (FIR), the upper bound given in (5) no longer holds because FIR models only form a subset of state-space models. A procedure called "Nehari Shuffle" is proposed in [9] to produce



(a) Reference channels: 3×3 with order/McMillan degree 15.

(b) Reference channels: 10×10 with order/McMillan degree 50.

Fig. 1. Upper bounds of minimum H_∞ channel approximation error w.r.t reference channels of size 3×3 and 10×10.

a low-order length-q FIR approximation to a MIMO linear system, which may or may not be FIR. If we express the transfer function of the original system as

$$\mathbf{G}(z) = \underbrace{\sum_{k=0}^{q-1} g_k z^{-k}}_{\triangleq \mathbf{G}^{head}(z)} + z^{-(q-1)} \underbrace{\sum_{l=1}^{\infty} g_{l+q-1} z^{-l}}_{\triangleq \mathbf{G}^{tail}(z)},$$

then an upper bound of the approximation error of the "Nehari Shuffle" procedure is given by the sum of all the Hankel singular values of the tail system $\mathbf{G}^{tail}(z)$,

$$\epsilon_{\infty,NS} \leq \sum_{i=1}^{n_t} \sigma_i(\mathbf{G}^{tail}), \tag{6}$$

where n_t is the order of $\mathbf{G}^{tail}(z)$.

Figure 1 shows the upper bound of the minimum H_∞ approximation error of state-space models, $\epsilon_{\infty,opt}$, and that of FIR models, $\epsilon_{\infty,NS}$, for both spatially white channels and spatially correlated channels. Both of the full-order state-space model and the full-length FIR model achieve zero H_∞ approximation error. However, more attention should be given to the performance of *low-order*, instead of full-order, state-space models and FIR models as discussed in the beginning of Section II-A.

Several observations on the performance of the low-order approximated models can be made. First, the number of low-order FIR models is significantly smaller than that of the low-order state-space models. This is because the order of an generic $p \times m$ length-L FIR model, one whose subchannels do not share any common zeros, is given by

$$min\{m, p\} \times (L-1).$$

Since the order of an FIR model can only be reduced by decreasing the length of the impulse responses, the step size of reducing the order of an FIR model is at least $min\{m, p\}$. On the other hand, the step size of reducing the order of an state-space model is 1. Therefore, state-space models provide us with many more choices for low-order approximation of the reference channel than FIR models.

Secondly, for a fixed order of which there exists an FIR model, the approximation error of state-space models is significantly smaller than that of FIR models. Furthermore, when we increase the dimension of the MIMO channel from 3×3 to 10×10, the spatial correlation in the reference channels also increases because of the fixed-length array baselines at both the mobile unit and the base station. In turn, the performance gap becomes more evident as the spatial correlation increases due to state-space models' ability to represent the MIMO channel as a whole instead of a group of separate impulse responses as in FIR models.

III. State-Space Adaptive Channel Estimation

In this section, we describe very briefly a recursive algorithm for estimating state-space channel models proposed in [3] and focus more on the performance comparison of the state-space-based channel estimator and the FIR-based Recursive Least Squares channel estimation algorithm. It is shown that the former exhibits marginally slower but comparable convergence rate to the latter while providing low-order channel models of higher quality.

A. Recursive state-space channel estimation

Subspace system identification (SSI) refers to a class of relatively recent algorithms, such as MOESP and N4SID, which apply input-output system identification methods to determine directly a (non-canonical) state-space realization of a system. These algorithms are available in the Matlab System Identification Toolbox.

The key idea of SSI methods is to estimate the extended observability matrix through the projection of future input-output data onto past input-output data based on the relationship

$$Y_{0,i,t} = \Gamma_i X_{0,t} + H_i U_{0,i,t} + N_{0,i,t},\ i > n, \tag{7}$$

where $U_{0,i,t}$, $Y_{0,i,t}$ and $N_{0,i,t}$ are Hankel matrices of the input, output and noise, respectively, with the form of

$$Y_{0,i,t} = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \cdots & \mathbf{y}_{t-i+1} \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{t-i+2} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_{i-1} & \mathbf{y}_i & \cdots & \mathbf{y}_t \end{bmatrix}. \tag{8}$$

$X_{0,t}$ is a matrix containing the channel state vectors

$$X_{0,t} = \begin{bmatrix} \mathbf{x}_0 & \cdots & \mathbf{x}_{t-i+1} \end{bmatrix}.$$

Γ_i and H_i are, respectively, the extended observability matrix and the matrix of impulse response coefficients,

$$\Gamma_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-2} \\ CA^{i-1} \end{bmatrix}, H_i = \begin{bmatrix} D & 0 & & & \cdots & 0 \\ CB & D & 0 & & \cdots & 0 \\ CAB & CB & D & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ CA^{i-2}B & & & & \cdots & D \end{bmatrix}.$$

The system matrices A and C are computed based on an estimate of the extended observability matrix, $\hat{\Gamma}_i$, which can be found by performing an LQ decomposition of the Hankel input-output data matrices as follows [10].

$$\begin{bmatrix} U_{0,i,t} \\ Y_{0,i,t} \end{bmatrix} = \begin{bmatrix} R_{11\ mi \times mi} & 0 \\ R_{21\ pi \times mi} & R_{22\ pi \times pi} \end{bmatrix} Q, \tag{9}$$

and

$$\hat{\Gamma}_i = R_{22}.$$

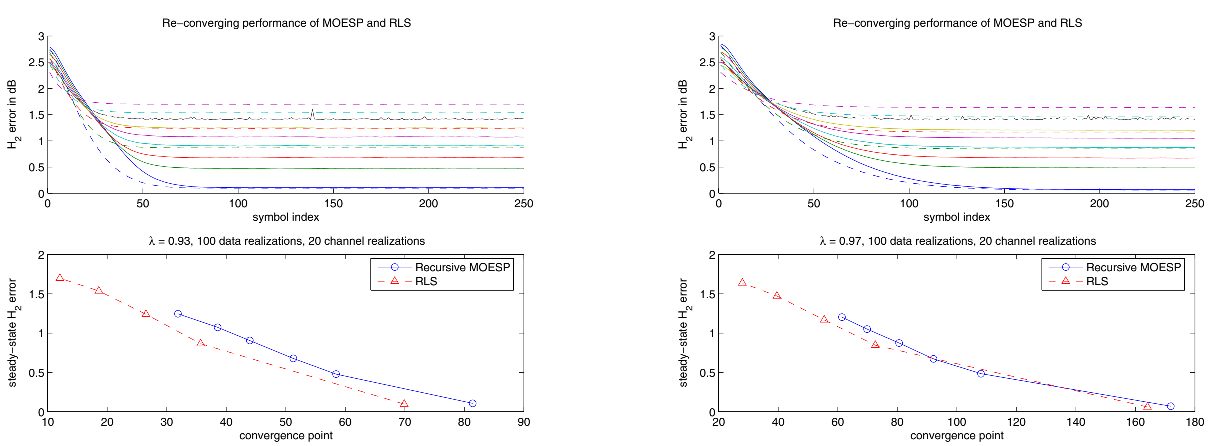
Then the associated matrices B and D for the realization determined by A and C may be computed from (7) using $\hat{\Gamma}_i$. See [10], [11], [12] for in-depth treatments of subspace system identification algorithms.

Recursive SSI algorithms have been developed to meet the need of estimating time-varying systems. A so-called recursive Ordinary MOESP scheme was proposed in [4], [5], whose key idea is to update the estimate of the extended observability matrix Γ_i with the newly received data using LQ factorization. In order to adapt the recursive MOESP algorithm for the specific problem of estimating MIMO wireless channels, [3] made further development by making modifications necessary for using non-contiguous training data and applying a new RLS-like exponential forgetting scheme on the training data to improve the adaptation rate.

As far as the stability of the estimated state-space models is concerned, most SSI algorithms do not guarantee that the estimated state-space models will be stable. There exist approaches to stabilizing state-space systems with the price of deviating from the original systems. We believe, however, that stability may not be a necessary requirement on the estimated models, since the ultimate goal of estimating channel models is to use them to design equalizers. It is the stability of the equalizers, as opposed to that of the channel estimates, that is desired.

B. *Performance of the adaptive state-space channel estimator*

The spatially correlated MIMO channels are generated in the same way as discussed in Section II-A. The training sequences are random binary sequences that are uncorrelated in both spatial dimension and time dimension. H_2 estimation error is only choosen over H_∞ because the former is numerically less expensive to compute. Both the SSI and FIR/RLS channel estimators were run until they reached a stationary value. Then, at time $t = 0$, the channel was changed and the identification methods allowed to reconverge. This is depicted in the upper plots of Figures 2(a) and 2(b) for 2×2 6-tap channels with SNR=20dB and illustrates the transient performance or learning rate of the identifiers. Here SNR is



(a) $\lambda = 0.93$, fast convergence.

(b) $\lambda = 0.97$, slow convergence.

Fig. 2. Estimation error of recursive MOESP and RLS for 2×2 spatially correlated channels.

defined to be ratio of the average power of transmitted symbols at each antenna over the average noise power at each antenna.

For the upper figures, the horizontal axis shows the number of training symbols sent after time $t = 0$. The vertical axis shows the H_2 error between the true channel and the model. There are separate curves for each different model order, with the dashed curves corresponding to the FIR models and the solid or dotted curves corresponding to state-space models. The items of interest are the steady-state error values, which indicate the modeling accuracy, and the learning or adaptation rate. The ordering of the curves shows that low-order models achieve poorer steady-state accuracy and faster learning. The curves themselves are the result of averaging over 20 channel (initial and final) realizations and, for each channel realization, 100 training symbol sequence realizations.

The lower figures are derived from the upper figures and plot one point for each model order and identification method, with triangles for the FIR/RLS method and circles for SSI/MOESP. The vertical axis is the steady–state modeling error and the horizontal axis shows the input symbol number from which the channel model error remains below 110% of the steady-state value. This latter figure gives a measure of convergence rate where smaller abscissa values imply faster adaptation.

The comparison of channel estimation performance between state-space models and FIR models, shown in Figures 2, is consistent with what is predicted by the upper bounds provided in Figure 1. Low-order state-space models exhibit significantly smaller steady-state estimation error than low-order FIR models. This performance gap becomes even larger as the spatial correlation of the channel, i.e. the channel redundancy, increases in large-dimensional MIMO channels (Figure 3).

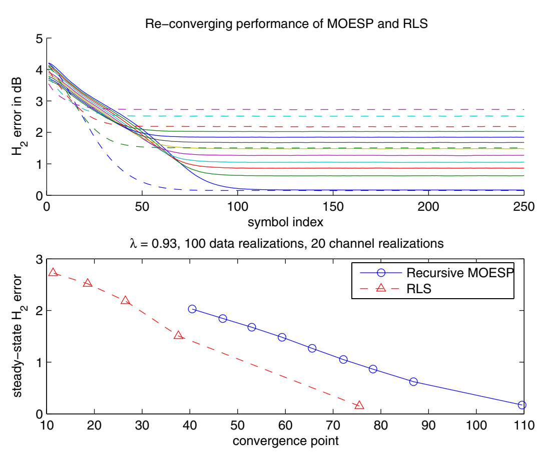


Fig. 3. Estimation error of recursive MOESP and RLS for 3×3 spatially correlated channels $\lambda = 0.93$.

The convergence rate of the recursive MOESP algorithm is in general slower than that of the RLS algorithm, but the difference stays in a comparable range. The faster convergence of RLS is to be expected because the (spatially and temporally) white training sequence is optimal for FIR model structure [12]. The demonstration that subspace system identification methods can deliver a high-performance (low-error) MIMO channel model with adaptation rate comparable to that of FIR-based RLS methods, validates their further consideration for the adaptive MIMO channel equalization problem.

IV. Conclusions

State space models are proposed to represent MIMO frequency-selective wireless channels with the motivation of better model approximation performance and more parsimonious parametrization. Upper bounds on the minimum H_∞ approximation error of both state-space models and FIR models are provided and compared to show that state-space models possess improved performance compared to more standard FIR models. The performance comparison of a state-space-based recursive channel estimator and the FIR-based RLS channel estimator shows that the former exhibits marginally slower but comparable convergence rate to the latter while providing low-order channel models of higher quality. This validates the consideration of state-space models for the adaptive MIMO channel modeling and equalization problem. Future work will be focused on the design and performance evaluation of state-space MIMO channel equalizer.

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