Abstract: Recent approaches to adaptive control with caution advocate the use of
the $\nu$-gap metric to restrict the permitted deviation between successive controllers
based on the achieved performance of the current controller. The aim is to limit
the magnitude of the controller adjustment in order to profit from the closed-
loop stability and performance guarantees associated with the $\nu$-gap metric. The
computation of the $\nu$-gap distance between two controllers involves two parts: the
checking of the satisfaction of a winding number condition, and the computation
of a frequency domain norm. In this paper we show by example that satisfaction
of the winding number condition is not a transitive relation and examine the
implications of this for cautious adaptive control.

Keywords: $\nu$-gap metric, winding number condition, caution

1. INTRODUCTION

The $\nu$-gap metric, $\delta_\nu(\cdot,\cdot)$, measures a distance
between two systems. In adaptive control it can
be used to measure the distance between two
controllers $C_0$, the current controller, and $C_1$, the
candidate next controller. The importance of this
measure is that if the $\nu$-gap metric satisfies
\begin{equation}
\delta_\nu(C_0, C_1) < b_{P,C_0},
\end{equation}
where $b_{P,C_0}$ is a stability margin, to be defined
shortly and computed or estimated from con-
troller $C_0$ applied to $P$, then plant $P$ is guaranteed
to be stabilized by controller $C_1$. Cautious adap-
tive control (Anderson and Gevers, 1998; Kammer
et al., 2000; Bitmead et al., 1999) uses this result
to restrict the permitted variation between suc-
cessive controllers during adaptation, and so profits
from stability and performance guarantees.

1 This work was supported the US National Science Foun-
dation under grant number ECS-0200449
1.1 Definitions and a guarantee

The $\nu$-gap metric (Vinnicombe, 1993; Zhou and Doyle, 1998) computation consists of two pieces: checking a winding number condition, and computing a frequency-domain norm.

**Definition 1.** (The Winding Number Condition; WNC). Two controllers, $C_0(s)$ and $C_1(s)$, satisfy the Winding Number Condition if

$$\det(I + C_1^*C_0)(j\omega) \neq 0, \forall \omega \text{ and }\ wno(I + C_1^*C_0) + \eta(C_0) - \eta(C_1) = 0,$$

where $wno(\cdot)$ indicates the winding number of the Nyquist diagram of the scalar transfer function evaluated on a contour enclosing the right-half-plane and indented along the imaginary axis to the right around any pure imaginary poles, and $\eta(C)$ is the number of open right-half-plane poles of $\det(C)$.

**Definition 2.** ($\kappa$ function). Given two commensurate transfer function matrices, $C_0(s)$ and $C_1(s)$,

$$\kappa(C_0, C_1) = \left\| (I + C_1C_1^*)^{-\frac{1}{2}}(C_1 - C_0)(I + C_0^*C_0)^{-\frac{1}{2}} \right\|_{\infty}.$$

**Definition 3.** (Vinnicombe’s $\nu$-gap metric).

$$\delta_\nu(C_0, C_1) = \begin{cases} \kappa(C_0, C_1), & \text{if the WNC holds} \\ 1, & \text{else} \end{cases}$$

The central importance for cautious adaptive control of the $\nu$-gap metric is captured in the following stability and performance guarantees. Firstly however, two definitions are needed.

**Definition 4.** (Generalized Sensitivity Function).

$$T(P, C) = \begin{pmatrix} P(I + CP)^{-1} C & P(I + CP)^{-1} \\ (I + CP)^{-1} C & (I + CP)^{-1} \end{pmatrix}.$$

**Definition 5.** (Generalized Stability Margin).

$$b_{P, C} = \begin{cases} \|T\|_{\infty}^{-1}, & \text{if } (P, C) \text{ is stable} \\ 0, & \text{else.} \end{cases}$$

**Theorem 1.** (Vinnicombe). Consider a plant $P$ and two controllers $C_0$ and $C_1$, with $C_0$ stabilizing $P$. Then the following results hold.

### Stability guarantee

$(P, C_1)$ is stable if

$$\delta_\nu(C_0, C_1) < b_{P, C_0}.$$  \hspace{1cm} (7)

### Performance guarantee

If $\delta_\nu(C_0, C_1) < 1$ then

$$\arcsin b_{P, C_1} \geq \arcsin b_{P, C_0} - \arcsin\delta_\nu(C_0, C_1),$$

and further, if $C_0$ and $C_1$ both stabilize $P$,

$$\delta_\nu(C_0, C_1) \leq \|T(P, C_0) - T(P, C_1)\|_{\infty}.$$ \hspace{1cm} (9)

2. CAUTIOUS ADAPTATION AND THE YK-HOMOTOPY

Anderson and Gevers (1998) proposed an approach to the selection of a new controller in which the inclusion of a $\nu$-gap limit is easily feasible. This is based on the use of the Youla-Kucera parametrization of stabilizing controllers. Here we develop this slightly by using a normalized coprime factorization for the definition of the homotopy.

**Definition 6.** (YK-homotopy). Suppose we are given:

- A plant model $P_0$ with normalized coprime factorizations $P_0 = XY^{-1} = \tilde{Y}^{-1}\tilde{X}$,
- $P_0$-stabilizing and $P$-stabilizing controller $C_0$ with right coprime factorization $C_0 = ND^{-1}$ satisfying $\tilde{Y}D + \tilde{X}N = I$,
- $P_0$-stabilizing candidate controller $C_1$ with right coprime factorization given by the Youla-Kucera parameter $Q, C_1 = (N - YQ)(D + XQ)^{-1}$.

Then we define the YK-homotopy of $P_0$-stabilizing controllers by

$$\{C_\alpha = (N - \alpha YQ)(D + \alpha XQ)^{-1}: \alpha \in [0, 1]\}.$$ \hspace{1cm} (10)

**Theorem 2.** For two controllers within the same YK-homotopy we have

$$\delta_\nu(C_{\alpha_1}, C_{\alpha_2}) \leq \|T(P_0, C_{\alpha_1}) - T(P_0, C_{\alpha_2})\|_{\infty} = |\alpha_1 - \alpha_2|\|Q\|_{\infty}. \hspace{1cm} (11)$$

**Proof:** The first inequality comes from Theorem 1. The second is derived as follows from...
\[ T(P_0, C_0) = \left( \frac{N}{D} \right) (\tilde{Y}D + \tilde{X}N)^{-1} \left( \tilde{X} \tilde{Y} \right) \]
\[ = \left( \frac{N}{D} \right) \left( \tilde{X} \tilde{Y} \right). \]

\[ T(P, C_{\alpha_1}) - T(P_0, C_{\alpha_2}) = (\alpha_1 - \alpha_2) \left( -\frac{Y}{X} \right) Q \left( \tilde{X} \tilde{Y} \right), \]

\[ \| T(P_0, C_{\alpha_1}) - T(P_0, C_{\alpha_2}) \|_\infty \]
\[ = |\alpha_1 - \alpha_2| \sup_{\omega} \lambda_{\max} \left[ \left( -\frac{Y}{X} \right) Q \left( \tilde{X} \tilde{Y} \right) \right. \]
\[ \left. \left( \tilde{X}^* \tilde{Y}^* \right) Q^* \left( -\frac{Y^* X^*}{X} \right) \right], \]
\[ = |\alpha_1 - \alpha_2| \|Q\|_\infty. \]

This establishes the following property.

**Corollary 1.** Within the YK-homotopy, \( \delta_\nu(C_0, C_\alpha) \) is continuous in \( \alpha \).

**Proof:** From the theorem above and the triangle inequality of the \( \nu \)-gap metric, we have
\[ |\delta_\nu(C_0, C_{\alpha + d\alpha}) - \delta_\nu(C_0, C_\alpha)| \leq \delta_\nu(C_{\alpha + d\alpha}, C_\alpha) \]
\[ \leq |d\alpha| \|Q\|_\infty. \]

We have the immediate corollary, which marginally extends an idea of Anderson and Gevers (1998).

**Corollary 2.** Given an arbitrary bound \( \epsilon > 0 \), it is always possible to choose a positive value \( \alpha_{\max} = \min \{1, \epsilon / \| Q \|_\infty\} \) such that \( \delta_\nu(C_0, C_\alpha) < \epsilon \) for all \( \alpha < \alpha_{\max} \).

This corollary may be used to develop cautious control adaptation by combining condition (7) with this bound on \( \alpha \). The result (11) further shows that the YK-homotopy provides a path between controllers \( C_0 \) and \( C_1 \) which is connected in the \( \nu \)-gap metric. It further provides a bound on the deviation of the designed closed-loop performance with model \( P_0 \) as \( \alpha \) is varied. Although this is less important than performance bounds on the performance with \( P \), which are captured by (8).

### 3. NONTRANSITIVITY OF THE WINDING NUMBER CONDITION

Satisfaction of the winding number condition (WNC) by two controllers is a reflexive and symmetric relation. Reflexivity follows since \( \det(I + C_0^* C_0)(j\omega) \) is a strictly positive function whose winding number is therefore zero. Thus the pair \((C_0, C_0)\) satisfies WNC. Symmetry follows from the conjugate-negation of \( \text{wnodet}(I + C_1^* C_0) \). So that if \((C_0, C_1)\) satisfies WNC, then so does \((C_1, C_0)\). Transitivity would make satisfaction of the winding number condition an equivalence relation.

Transitivity of the WNC would validate the following implication.
\[ [(C_0, C_1) \text{ satisfies WNC}] \land [(C_1, C_2) \text{ satisfies WNC}] \]
\[ \implies [(C_0, C_2) \text{ satisfies WNC}]. \]

We shall shortly demonstrate by example that this is false. However, were it to be true, then one would have from Corollary 2 that all controllers in the YK-homotopy would satisfy pairwise the winding number condition. Then the search for guaranteed \( P \)-stabilizing controllers would need only to consider the computation of the frequency norm \( \kappa(C_0, C_\alpha) \) among the candidate controllers. As will be demonstrated in the example, the \( \kappa \) function can return close to zero as one explores the YK-homotopy for larger values of \( \alpha \) than those given by Corollary 2. The fact that we need separately to verify WNC is a reflection of the coarseness of the \( \nu \)-gap metric.

#### 3.1 Example

We borrow an example from (Blondel et al., 1997), in which two dramatically different plant models are stabilized by the same constant controller, yielding very closely similar closed-loop responses. We consider the dual of the example.
Fig. 1. The $\nu$-gap distance between $C_0$ and $C_\alpha$ (solid curve) and between $C_1$ and $C_\alpha$ as a function of $\alpha$.

$$P(s) = 5.918, \quad X(s) = \bar{X}(s) = 0.9860, \quad Y(s) = \bar{Y}(s) = 0.1666,$$

$$C_0(s) = \frac{(s-1)}{(s-2)(s-3)}, \quad D(s) = \bar{D}(s) = \frac{(s-2)(s-3)}{0.1666s^2 + 0.1530s + 0.0137},$$

$$N(s) = \bar{N}(s) = \frac{0.1666s^2 + 0.1530s + 0.0137}{(s-1)}, \quad C_1(s) = \frac{-1.22}{s + 7.32},$$

$$Q(s) = \frac{2.222s^2 + 0.22s}{0.0278s^3 + 0.0283s^2 + 0.048s + 0.0002}. $$

These transfer functions satisfy the conditions required for us to develop the YK-homotopy of controllers, $\{C_\alpha(s)\}$. Figure 1 shows two curves. The solid curve depicts $\delta_\nu(C_0, C_\alpha)$ versus $\alpha$, and the dashed curve shows $\delta_\nu(C_1, C_\alpha)$ versus $\alpha$. A zoomed version of the region in the neighborhood of $\alpha = 0.383$ is shown in Figure 2. In these figures, a value of $\delta_\nu$ less than one indicates that the winding number condition is satisfied by the two transfer functions. Thus from Figure 2, it is apparent that $(C_0, C_{0.383})$ and $(C_1, C_{0.383})$ satisfy WNC. A simple examination shows that $(C_0, C_1)$ does not satisfy WNC. This establishes by example that WNC is not transitive.

Figure 3 shows the plot of $\kappa(C_0, C_\alpha)$ as a function of $\alpha$. For small values of $\alpha$ it follows the $\delta_\nu(C_0, C_\alpha)$ curve, of course. However, for larger $\alpha$ the function $\kappa$ fails to indicate the dissatisfaction of WNC.

From an adaptive control perspective, this example is of interest, because it highlights the importance of checking the WNC as part of the evaluation of the guarantees of closed-loop performance via the $\nu$-gap metric. In this case, it also demonstrates the conservativeness of some of the $\nu$-gap results, because $C_\alpha$ stabilizes $P$ for any value of $\alpha$ even outside $[0,1]$.

4. CONCLUSION

We have presented and analyzed the Winding Number Condition component of the $\nu$-gap metric and its role in the study of cautious adaptation. While the results might not be surprising in hindsight, they do illustrate some interesting features of the $\nu$-gap metric, such as continuity and
connectedness of the metric space of stabilizing controllers.

REFERENCES