IDENTIFICATION OF RESONANT SYSTEMS USING PERIODIC MULTIPLICATIVE REFERENCE SIGNALS

Wayne J. Dunstan * **Robert R. Bitmead** *,[1]

* *Department of Mechanical & Aerospace Engineering, University of California San Diego, La Jolla CA 92093-0411, USA.*

Abstract: System identification of the forward path of a linear unity feedback system, which is almost unstable, is considered through the application of a periodic multiplicative reference signal. Motivation for this problem comes from the need to identify the linear acoustics of a combustion process on the verge of instability. The tools of cyclostationary signal analysis provide an entré to develop methods for understanding the problem and for determining requisite properties of the reference signal.

Keywords: Resonant systems, Excitation, Periodic Systems, Cyclostationary.

1. INTRODUCTION

The motivation for this paper derives from a practical problem occurring in combustion chambers of turbomachinery. It is desirable from pollution and economic reasons to run these engines at low fuel-to-air ratios. Operating engines in these ranges can produce strong periodic instabilities in the 100-1000Hz range, believed to stem from the nonlinear interaction between chamber pressure and the heat release rate.

An experimental rig used to observe these instabilities is illustrated in Figure 1. For low fuel-to-air ratio, the resulting spectra of pressure and heat release signals from this experiment demonstrate very strong tonal components consistent with the existence of a limit cycle. As this instability is approached however, we see linear behavior tending to increased resonance (Murray *et al.*, 1998; Dunstan and Bitmead, 2002; Dunstan *et al.*, 2001).

A block diagram of the physical model for this system is shown in Figure 2, where $v(t)$ is a driving noise input, $G(z)$ is the transfer function of interest (believed to consist of linear acoustics plus delay elements), and the memoryless saturation nonlinearity is present in the feedback path. In the linear regime of operation, the saturation is not operative but the closed loop progressively approaches resonance. The ultimate objective is to ameliorate these instabilities using a control design from a low complexity model, which requires identifying $G(z)$. Before the onset of instability the nonlinearity can be replaced by a static gain and the system may be treated using linear tools. In our analysis, the reference $r(t)$ will be periodic and

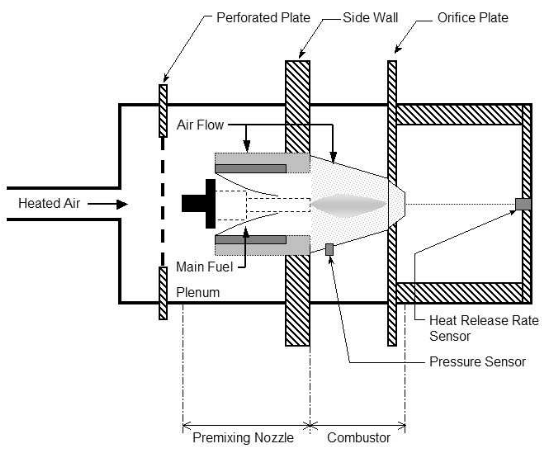

Fig. 1. Single Nozzle Rig experimental setup

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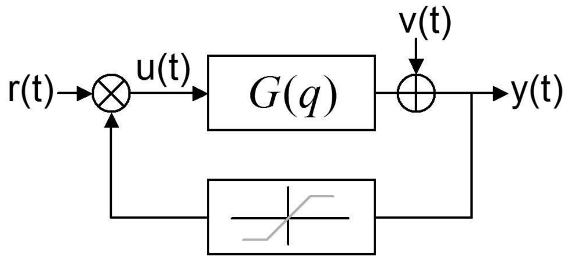

Fig. 2. Combustion instability model.

enter the loop multiplicatively, as shown in Figure 2. Signal notations are as defined by this figure.

The system identification task is then to create an estimate of $G(z)$, denoted $\hat{G}(z)$, with $r(t)$ a known periodic signal and the unity feedback closed loop $[1-G(z)]^{-1}$ highly resonant. The implications of this are,
(a) the periodic multiplicative reference signal results in a linear time varying system with cyclostationary signals, and,
(b) the highly resonant system is challenging to identify using standard closed loop techniques.

1.1 Limitations in resonant system estimation

Considering the case when $r(t) = 1$, the system reduces to a stationary noise-driven system with transfer function description,

$$y(t) = [1 - G(z)]^{-1}\, v(t). \tag{1}$$

An Auto-Regressive (AR) model,

$$y(t) = \frac{1}{A(z)} v(t)$$

of arbitrary order may be fitted to output data, $\{y(t)\}$, from this system. The resulting estimate, $\hat{A}(z)$ and therefore $\hat{G}(z) = 1 - \hat{A}(z)$, will be biased due to the inability of the AR model to capture adequately the noise dynamics, $v(t)$, added to the signal leaving $G(z)$ (Quinn and Fernandes, 1991). Further, as the resonance of the closed-loop becomes stronger, $y(t)$ becomes more strongly tonal and the bias worsens. For the computed model $\hat{G}(z)$ the lowest bias will be where $G(e^{j\omega})$ is close to 1, that is, where the signal spectrum is most persistently exciting.

Equation (1) implies the following the power spectral density (PSD) relationship.

$$\Phi_y(\omega) = [I - G(e^{j\omega})]^{-1}\ \Phi_v(\omega)\ [I - G(e^{-j\omega})]^{-T}$$

where $\Phi_y(\omega)$ and $\Phi_v(\omega)$ are the PSD's of $y(t)$, $v(t)$ respectively.

Example 1. Consider the system in (1) with,

$$G(z) = \frac{0.1}{z^2 + 2\alpha cos(\beta) z + \alpha^2}$$

where $\alpha = 0.9$ and $\beta = \pi/7$.

Fitting a 20^{th} order AR model to the $\{y(t)\}$ data results in the comparative fit shown in Figure 3. Notice

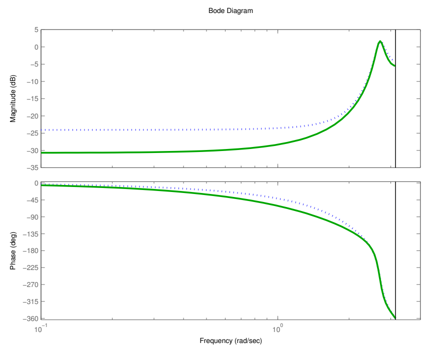

Fig. 3. Comparison of $G(z)$ (solid) and $\hat{G}(z)$ (dotted) for Example 1

the low bias in regions where $G(e^{j\omega})$ is close to 1, and the large bias elsewhere. This illustrates the problems which occur when $\Phi_y(\omega)$ contains strong tonal components.

Our aim is to quantify the improvements to the transfer function estimation problem introduced by a multiplicative reference signal.

The paper proceeds as follows:
- Establish that the signals are cyclostationary when a periodic multiplicative reference signal is used.
- Introduce tools of cyclostationary signal analysis based upon the lifting to a stationary vector process.
- Consider Least Squares parameter estimation of the stationary vector process.
- Determine properties of the reference signal to guarantee identifiability of $G(z)$.

2. CYCLOSTATIONARY SYSTEM ANALYSIS

Suppose that the multiplicative reference $r(t)$ is a deterministic periodic signal of period N and denote by z the z-transform variable and by q the forward-shift operator. Then the closed-loop system in Figure 2 is described by

$$y(t) = G(q)u(t) + v(t), \tag{2}$$
$$u(t) = r(t)y(t). \tag{3}$$

Suppose further that this stable. Then $y(t)$ is the output of a stable, linear, periodically time-varying, (LPTV) system driven by noise and thus is asymptotically a cyclostationary process (Gardner, 1994; Hale, 1969).

2.1 Lifting

The scalar LPTV system in (2) and (3) is transformed to a LTI vector system, of dimension N, using an isomorphism called *lifting* (Khargonekar *et al.*, 1985). Denote the unit delay operator q^{-1} and the N-step delay operator $\zeta = q^{-N}$, and the equivalent transform variable

$$\sigma = z^{-N}.$$

We identify the time variable as

$$t = kN + \ell.$$

Variable t is in the original single-sample domain and variable k is in the lifted N-sample domain. Define scalar signals $y_\ell(k) = y(kN+\ell)$ for $\ell = 0, \ldots, N-1$ and the *lifted* N-vector signal

$$y_k = \begin{bmatrix} y(kN+0) \\ y(kN+1) \\ \vdots \\ y(kN+N-1) \end{bmatrix}. \quad (4)$$

Similarly define the σ-transforms

$$\begin{aligned} Y_\ell(\sigma) &= \sum_{j=0}^{\infty} y_\ell(j)\sigma^j \\ &= \sum_{j=0}^{\infty} y(jN+\ell)\sigma^j, \quad \text{for} \;\; \ell = 0, \ldots, N-1. \end{aligned}$$

Then define

$$Y(\sigma) = \sum_{k=0}^{\infty} y_k \sigma^k = \begin{bmatrix} Y_0(\sigma) \\ Y_1(\sigma) \\ \vdots \\ Y_{N-1}(\sigma) \end{bmatrix}.$$

This is the σ-transform of the lifted process.

The N-vector lifted process y_k is a stationary stochastic process in time-variable k. This is the lifted form of the cyclostationary scalar process $y(t)$. Clearly lifting is an isomorphism, which may be applied conformably to $u(t)$ and $v(t)$ to yield lifted N-vector sequences, u_k and v_k. Using these N-vector sequences and the identity $\zeta = q^{-N}$, the relations of (2) and (3) may be rewritten

$$y_k = \mathcal{G}(\zeta)u_k + v_k, \quad (5)$$

$$u_k = Ry_k. \quad (6)$$

To determine $\mathcal{G}$ and R, write $G(\zeta) = \sum_{i=1}^{\infty} g_i\zeta^{-i}$ so that g_i are the impulse response parameters of G then matrix $\mathcal{G}(\sigma)$ is Toeplitz and square.

$$\mathcal{G}(\zeta) = \begin{bmatrix} \tilde{g}_0(\zeta) & \zeta\tilde{g}_{N-1}(\zeta) & \cdots & \zeta\tilde{g}_1(\zeta) \\ \tilde{g}_1(\zeta) & \tilde{g}_0(\zeta) & \cdots & \zeta\tilde{g}_2(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{g}_{N-1}(\zeta) & \tilde{g}_{N-2}(\zeta) & \cdots & \tilde{g}_0(\zeta) \end{bmatrix},$$

where

$$\tilde{g}_i(\zeta) = (g_i \; + \; \zeta^1 g_{N+i} \; + \; \zeta^2 g_{2N+i} \; + \ldots)$$

and,

$$R = \begin{bmatrix} r_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N-1} \end{bmatrix}.$$

Equation (5) and (6) imply the following the PSD relationship.

$$\Phi_y(\Omega) = [I - \mathcal{G}(e^{j\Omega})R]^{-1} \, \Phi_v(\Omega) [I - \mathcal{G}(e^{-j\Omega})R]^{-T} \quad (7)$$

where $\Phi_y(\Omega)$ and $\Phi_v(\Omega)$ are the PSD's of y_k, v_k respectively in the lifted frequency domain Ω. The mapping in (7),

$$\boxed{[\Phi_v(\Omega), G(z), r(t)] \Rightarrow \Phi_y(\Omega)}$$

alludes to the influence of the noise spectrum and reference signal $r(t)$ on the spectrum of y_k.

This section establishes an isomorphism under which the scalar cyclostationary system is replaced by an N-vector stationary lifted system with a matrix transfer function $\mathcal{G}(\sigma)$ and a relationship between the spectrum of this vector signal and the reference and noise signals.

3. LEAST SQUARES PARAMETRIZATION AND ESTIMATION

We develop a Least Squares (LS) estimator for vector systems, appropriate for application to the estimation of the parameters of this N-vector stationary y_k process. Write the first row of (5),

$$\begin{aligned} y(kN) = \tilde{g}_0(\zeta)\, u(kN) + \zeta\tilde{g}_{N-1}(\zeta)\, u(kN+1) + \cdots \\ + \zeta\tilde{g}_1(\zeta)\, u(kN+N-1) + v(kN) \quad (8) \end{aligned}$$

Observation 1. All $\tilde{g}_i(\zeta)$, $i{=}0..N{-}1$ have the same scalar denominator polynomial $A(\zeta)$ of degree m.

Write (8) in polynomial form (note the reordering of the $u(kN+1) \ldots u(kN+N-1)$ terms),

$$\begin{aligned} A(\zeta)y(kN) = B_0(\zeta)\, u(kN) + B_1(\zeta)\, \zeta u(kN+N-1)(\zeta) + \cdots \\ + B_{N-1}(\zeta)\, \zeta u(kN+1) + \varepsilon(kN) \end{aligned}$$

where,

$$\begin{aligned} \varepsilon(kN) &= A(\zeta)v(kN), \\ A(\zeta) &= 1 + a_1\zeta + \ldots + a_m\zeta^m, \\ B_i(\zeta) &= b_{i,0} + b_{i,1}\zeta + \ldots + b_{i,m}\zeta^m \qquad i{=}0..N{-}1, \end{aligned}$$

and it is assumed that all $B_i(\zeta)$ polynomials also have degree m.

This is a scalar AutoRegressive with an eXogenous input (ARX) model, for which a parametrization can be written,

$$\begin{aligned}
y(kN) = & -a_1 y((k-1)N) - \cdots - a_m y((k-m)N) \\
& + b_{0,0} u(kN) + \cdots + b_{0,m}\, u((k-m)N) \\
& + b_{1,0} u((kN-1) + \cdots + b_{1,m}\, u((k-m)N-1) \\
& + b_{2,0} u((kN-2)) + \cdots + b_{2,m}\, u((k-m)N-2) \\
& \vdots \\
& + b_{N-1,0} u((kN-(N-1)) + \cdots + b_{N-1,m} u((k-m)N-(N-1) \\
& + \varepsilon(kN)
\end{aligned}$$

Writing the other rows in a similar way yields,

$$y_k = \Psi_k^T \, \theta + \varepsilon_k, \tag{9}$$

where the $N \times [(N+1)m - 1]$ regressor and $N \times N$ reference matrices,

$$\Psi_k^T = [\, \bar{\mathcal{Y}}_0 \mid R_0 \mathcal{Y}_0 \mid R_1 \mathcal{Y}_1 \mid \cdots \mid R_{N-1} \mathcal{Y}_{N-1} \,] \tag{10}$$

and,

$$\mathcal{Y}_i = \begin{bmatrix} y((k-0)N+0-i) & \cdots & y((k-m)N+0-i) \\ y((k-0)N+1-i) & & y((k-m)N+1-i) \\ \vdots & & \vdots \\ y((k-0)N+N-1-i) & \cdots & y((k-m)N+N-1-i) \end{bmatrix}$$

$$R_i = \begin{bmatrix} r_{0-i} & 0 & 0 & \cdots & 0 \\ 0 & r_{1-i} & 0 & \cdots & 0 \\ 0 & 0 & r_{2-i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N-1-i} \end{bmatrix}$$

where for $x < 0$ $r_x = r_{N-x}$, and,

$\bar{\mathcal{Y}}_0 = -\mathcal{Y}_0$ with first column removed.

$$\theta = [\, a_1 \cdots a_m \mid b_{0,0} \cdots b_{0,m} \mid b_{1,0} \cdots b_{N-1,m} \,]^T$$

$$\varepsilon_k = \begin{bmatrix} \varepsilon(kN) \\ \varepsilon(kN+1) \\ \vdots \\ \varepsilon(kN+N-1) \end{bmatrix}.$$

The structure of $\mathcal{Y}_i$ is summarized:

- The elements going down the rows represent increasing samples along each period of $r(t)$.
- Stepping across the columns represents an N-step time shift in the signals.

Defining a scalar cost function for $M = D \times N$ data,

$$V_M = \sum_{k=0}^{D-1} (y_k - \Psi_k^T \theta)^T (y_k - \Psi_k^T \theta).$$

Note that this lifted form differs from the regular least squares formulation through the appearance of an N-vector measurement y_k and an $N \times [(N+1)m - 1]$ regressor matrix Φ_k. The minimum is given by the standard LS solution structure with increased dimension elements,

$$\hat{\theta}_M = \left[\frac{1}{D} \sum_{k=0}^{D-1} \Psi_k \Psi_k^T \right]^{-1} \frac{1}{D} \sum_{k=0}^{D-1} \Psi_k y_k. \tag{11}$$

Equation (11) forms the link,

$$\boxed{\mathbf{E}\left[\Psi \Psi^T\right] \Rightarrow \hat{\theta}.}$$

We note this parametrization of $\mathcal{G}(\sigma)$ is isomorphic to a parametrization of $G(z)$. Identifiability of $\mathcal{G}$ relies on properties of the stationary N-vector signal y_k which are inherited (or descended) from the properties of cyclostationary $y(t)$, whose analysis lacks tools.

4. IDENTIFIABILITY

Identifiability imposes conditions on the input and output signals to ensure an estimate $\hat{G}(z)$ exists and is unique. In this context, we consider the identifiability of $\mathcal{G}(z)$ from the lifted N-vector LS problem of the previous section. The lifting admits appeal to tools from stationary signals analysis, which are inaccessible for the original cyclostationary processes. Further, (7) establishes the spectral/correlation properties of the N-vector stationary signals.

Theorem 1. Parameter θ in (9) is identifiable if

$$\mathcal{I} = \frac{1}{D} \sum_{k=0}^{D-1} \Psi_k \Psi_k^T \tag{12}$$

has full rank.

The identifiability matrix, $\mathcal{I}$, for (10) is,

$$\mathcal{I} = \frac{1}{D} \times \begin{bmatrix} \bar{\mathcal{Y}}_0^T \bar{\mathcal{Y}}_0 & \bar{\mathcal{Y}}_0^T R_0 \mathcal{Y}_0 & \cdots & \bar{\mathcal{Y}}_0^T R_{D-1} \mathcal{Y}_{N-1} \\ \mathcal{Y}_0^T R_0 \bar{\mathcal{Y}}_0 & \mathcal{Y}_0^T R_0^T R_0 \mathcal{Y}_0 & \cdots & \mathcal{Y}_0^T R_0^T R_{N-1} \mathcal{Y}_{N-1} \\ \mathcal{Y}_1^T R_1 \bar{\mathcal{Y}}_0 & \mathcal{Y}_1^T R_1^T R_0 \mathcal{Y}_0 & \cdots & \mathcal{Y}_1^T R_1^T R_{N-1} \mathcal{Y}_{N-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{Y}_{N-1}^T R_{N-1} \bar{\mathcal{Y}}_0 & \mathcal{Y}_{N-1}^T R_{N-1}^T R_0 \mathcal{Y}_0 & \cdots & \mathcal{Y}_{N-1}^T R_{N-1}^T R_{N-1} \mathcal{Y}_{N-1} \end{bmatrix}$$

This Toeplitz, symmetric, nonnegative definite matrix contains elements which depend on the correlation properties of y_k and $r(t)$. This can be exploited to establish an alternative identifiability condition. This, in turn, provides conditions necessary for determining how $r(t)$ affects the identifiability of θ and therefore $G(z)$.

Define the auto-correlation function of $y(t)$

$$\mathcal{R}_y(t, \tau) = E(y(t) y(t - \tau)).$$

This is both t- and τ-dependent and is N-periodic in t (Gardner, 1994). We now consider the relationship between the elements of $\mathcal{I}$, $\mathcal{R}_y(t,\tau)$ and $r(t)$ by computing representative expansions. The sub-blocks of $\mathcal{I}$ are Toeplitz matrices comprised of combinations of $\mathcal{R}_y(t,\tau)$ and r_i.

Elements of the average of $\bar{\mathcal{Y}}_0^T\bar{\mathcal{Y}}_0$ have composition,

$$
\begin{aligned}
(1{,}1) &= \mathcal{R}_y(1{,}0)+\mathcal{R}_y(2{,}0)+\cdots+\mathcal{R}_y(N-1{,}0)\\
(1{,}2) &= \mathcal{R}_y(1{,}N)+\mathcal{R}_y(2{,}N)+\cdots+\mathcal{R}_y(N-1{,}N)\\
(1{,}N) &= \mathcal{R}_y(1{,}mN)+\mathcal{R}_y(2{,}mN)+\cdots+\mathcal{R}_y(N-1{,}mN)
\end{aligned}
$$

Elements of the average of $\mathcal{Y}_0^T R_0^T R_0 \mathcal{Y}_0$ have composition:

$$
\begin{aligned}
(1{,}1) &= r_0^2\mathcal{R}_y(0{,}0)+r_1^2\mathcal{R}_y(1{,}0)+\cdots+r_{N-1}^2\mathcal{R}_y(N-1{,}0)\\
(1{,}2) &= r_0^2\mathcal{R}_y(0{,}N)+r_1^2\mathcal{R}_y(1{,}N)+\cdots+r_{N-1}^2\mathcal{R}_y(N-1{,}N)\\
(1{,}N) &= r_0^2\mathcal{R}_y(0{,}mN)+r_1^2\mathcal{R}_y(1{,}mN)+\cdots+r_{N-1}^2\mathcal{R}_y(N-1{,}mN)
\end{aligned}
$$

Elements of the average of $\bar{\mathcal{Y}}_1^T R_1^T R_2 \bar{\mathcal{Y}}_2$ have composition,

$$
\begin{aligned}
(1{,}1) &= r_{N-1}r_{N-2}\mathcal{R}_y(N-1{,}1)\\
&\qquad +r_0r_{N-1}\mathcal{R}_y(0{,}1)+\cdots+r_1r_2\mathcal{R}_y(1{,}1)\\
(1{,}2) &= r_{N-1}r_{N-2}\mathcal{R}_y(N-1{,}N-1)\\
&\qquad +r_0r_{N-1}\mathcal{R}_y(0{,}N-1)+\cdots+r_1r_2\mathcal{R}_y(1{,}N-1)\\
(1{,}N) &= r_{N-1}r_{N-2}\mathcal{R}_y(N-1{,}mN-1)\\
&\qquad +r_0r_{N-1}\mathcal{R}_y(0{,}mN-1)+\cdots+r_1r_2\mathcal{R}_y(1{,}mN-1)
\end{aligned}
$$

Since the matrices are Toeplitz only the first row and column need to be calculated.

We have shown by exhausting example that the identifiability conditions for the forward-path plant $G(z)$ depend explicitly on the autocorrelation of $y(t)$ and the reference $r(t)$, which in turn are connected through (7). That is,

$$\boxed{\Phi_y(\Omega) \Rightarrow \mathcal{R}_y(t,\tau), r(t) \Rightarrow \mathcal{I}.}$$

5. ROAD MAP

This paper has investigated the system identification of a linear unity feedback system, which is almost unstable, through the application of a periodic multiplicative reference signal. Tools have been developed and applied to move toward a solution in which one can determine the identifiability properties of the forward path transfer function $G(z)$ through the appropriate selection of periodic reference signal $r(t)$.

Tools of cyclostationary signal analysis have been used to create,

PSD relationship

$$[\Phi_v(\Omega), G(z), r(t)] \Rightarrow \Phi_y(\Omega)$$

Vector Least Squares Parametrization

$$\mathbf{E}\left[\Psi\Psi^T\right] \Leftrightarrow \hat{\theta}$$

and to explore,

Identifiability Condition

$$\Phi_y(\Omega) \Rightarrow \mathcal{R}_y(t,\tau), r(t) \Rightarrow \mathcal{I}$$

for which analysis continues on the link between the Identifiability Matrix and the Correlation Matrices,

$$\mathcal{I} \Leftrightarrow \mathbf{E}\left[\Psi\Psi^T\right]$$

Some representative expansions of $\mathcal{I}$ begin to unveil the how the structure of $r(t)$ affects the rank properties of the $\mathcal{Y}_i^T\mathcal{Y}_j$ matrices. These matrices are composed of correlation matrices, $\mathcal{R}_y(t,\tau)$, which are indicative of the shape of $\Phi_y(\Omega)$. This path allows us to study the influence of $r(t)$ on resonant system identification. It is expected that the multiplicative effect of $r(t)$ will introduce sidebands into $\Phi_y(\Omega)$, which are observable in $\mathcal{R}_y(t,\tau)$.

Further work will link properties of the correlation matrices, $\mathcal{R}_y(t,\tau)$ which ultimately will show how to design $r(t)$ to improve the identification of resonant systems. How does this work?

Resonance and Sidebands

Resonance of the unexcited ($r(t) = 1$) loop, which for the practical combustion instability problem represents the onset of limit cycling, is associated with there existing a single base-rate frequency ω_0 such that

$$G(e^{j\omega_0}) \approx 1. \tag{13}$$

In signal terms, this corresponds to the appearance of strongly dominant single frequency in the closed-loop scalar signal spectrum $\Phi_y(\omega)$.

In the unexcited lifted framework (13) translates to the property that

$$\det\left[I - \mathcal{G}(e^{-jN\omega_0})\right] \approx 0, \tag{14}$$

or $\lambda_i\left(\mathcal{G}(e^{-jN\omega_0})\right) \approx 1$. Analysis of (13) and the structure of $\mathcal{G}$ indicates that a left eigenvector associated with this eigenvalue of one is

$$v_0 = \left[1\ e^{j\omega_0}\ \ldots\ e^{j(N-1)\omega_0}.\right] \tag{15}$$

With the introduction of non-constant N-periodic $r(t)$ composed of a constant plus small sinusoidal variation

$$r(t) = 1 + \beta\cos(\nu t),\quad \beta \ll 1,\quad \nu = \frac{\alpha}{N},\ \alpha \in \mathbb{N}$$

this eigenvector is perturbed by diagonal matrix R to two others with eigenvalues close to one and within a β-neighborhood of v_0. The signal frequencies associated with these perturbed eigenvalues are $\omega_0 + \nu$ and $\omega_0 - \nu$. This is the appearance of sidebands in the closed-loop signals. Note that the original resonant signal at frequency ω_0 remains.

Improved Identifiability
The splitting of the dominant closed-loop signal to yield persistent frequency content at new adjacent frequencies through sideband formation has an immediate effect on the identifiability of $G(z)$. As was demonstrated in the introduction via example, the estimation of G is accurate and unbiased at frequencies where the signal-to-noise ratio is high in closed loop. Sidebands extend the region of identifiability of G from just its frequency response at ω_0 (which is fully captured by just two real parameters) to include also the frequency response at $\omega_0 \pm \nu$, thereby extending the information content to six real parameters. This permits the improved estimation of the forward path in the resonant system.

This conceptual analysis needs to be extended to a quantified treatment of the properties of the identifiability matrix $\mathcal{I}$. This remains to do. Further work will focus on the specific design of the signal $r(t)$.

Combustion Instability Modeling for Control
In the practical example of combustion instability, the multiplicative reference is achieved through fuel modulation - a small proportion of the fuel flow is modulated at frequency ν. This can be a difficult actuation problem except at relatively low frequencies $\nu \ll \omega_0$. However, this is precisely where the excitation can have the desired effect of exposing the frequency response of G in the neighborhood of the resonance ω_0.

A control solution to the combustion instability would most likely rely on the correct phasing of a small fuel flow at the resonant frequency to assist in quenching the instability (Banaszuk *et al.*, 2003; Zhang *et al.*, 1998). To achieve this diminution of the instability with a model-based controller, accurate phase information is needed in the neighborhood of ω_0. Thus, low-frequency reference excitation is an approach well-suited to this practical problem.

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