

# Robust Constrained Model Predictive Control using Closed-loop Prediction <sup>1</sup>

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## Abstract

This paper proposes a quadratic programming (QP) approach to robust MPC for constrained linear systems having both model uncertainties and bounded disturbances. To this end, we construct an additional comparison model for worst-case analysis based on a robust control Lyapunov function (RCLF) for the unconstrained system (not necessarily an RCLF in the presence of constraints). By using this comparison model, we transform the given robust MPC problem to a nominal one without uncertain terms. This comparison model also enables us to derive a terminal condition for ensuring the robust stability of the closed-loop. Since this terminal condition is described by linear constraints, the control optimization can be reduced to a QP problem.

## 1 Introduction

Model predictive control (MPC), which determines online the control input by solving a finite horizon open-loop control optimization problem, is one of the few tractable ways to handle design problems having input and state constraints. (See [1] for an overview.) While most constrained MPC methods have been developed without taking account of model uncertainties, several studies of robust constrained MPC for uncertain systems have been reported [2–12]. This paper is particularly related to the state-space approaches [7–12] rather than the early works based on impulse response models [3–6].

One of the main drawbacks of many robust MPC methods for practical use is the complexity of on-line optimal control problems. Because of uncertain terms in prediction models, the optimal control problem at each time step is typically described as a min-max optimization problem subject to the condition that given con-

straints are satisfied for all possible uncertainties. Another limitation is that control design is so conservative that optimization problems are easily led to infeasibility, especially for unstable systems. This conservativeness is caused by the fact that a single open-loop control trajectory is determined to deal with all possible uncertainties. Although the importance of Feedback MPC, which determines feedback control laws rather than open-loop control trajectories, has been suggested recently [1, 9, 10], the control optimization required is basically much more complex than that for open-loop MPC.

One way to decrease complexity of robust MPC methods is to modify optimal control problems by using appropriate sufficient conditions which imply the given constraints are satisfied for all possible uncertainties. Based on such modified control problems, the following two types of approaches to robust MPC have been studied recently [11, 12].

Kothare *et al.* [11] proposes a robust MPC method for constrained linear systems with structured or polytopic uncertainties, which obtains online a state feedback gain based on a convex optimization using linear matrix inequalities (LMIs). An interesting feature of this approach is that, instead of predicting future state trajectories, a state-invariant set is obtained at every time step. In order to reduce the control problem to convex optimization, the control and state constraints originally given by using the state trajectories are transformed into sufficient conditions via the state-invariant set. Therefore, while those sufficient conditions save much computational burden, the control design could be conservative even for small uncertainties. Another remarkable point is that this method can handle a more general class of model uncertainties than most other MPC methods, although it is difficult to deal with disturbances explicitly.

Bemporad [12] discusses a robust MPC method using closed loop prediction made up of both feedback and open-loop control, which is probably the most related

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study to our paper. For linear systems with bounded disturbances, it is shown that the feedback part helps to decrease the conservativeness for disturbances, whereas the constraints are satisfied by the open-loop control. Another important feature is that the given constraints including disturbance terms are replaced by their sufficient conditions described by upper bounds of disturbances. As a result, the given robust MPC problem is transformed into a nominal problem without disturbance terms. This method however has difficulty in handling more general uncertainties, of which effects typically depend on unknown future state trajectories. Another limitation is that stability analysis in terms of a given finite horizon is difficult. It might be required to introduce an appropriate terminal condition in the finite-horizon control problem to overcome this problem.

In this paper, we propose a new robust MPC method based on closed-loop prediction for constrained linear systems with model uncertainties and bounded disturbances. In order to predict the worst-case effects of the model uncertainties as well as disturbances, we introduce a comparison model based on a robust control Lyapunov function (RCLF) [13] for the unconstrained system, which is not necessarily an RCLF in the presence of constraints. By using this comparison model, we transform the given robust MPC problem to a nominal one without uncertain terms. This comparison model also enables us to derive a terminal condition for ensuring the robust stability of the closed-loop. Furthermore, since this terminal condition is described as linear constraints, the control optimization problem for the proposed method can be solved by standard quadratic programming (QP) methods.

## 2 Problem formulation

Let  $\|x\|$  and  $\|x\|_p$  denote the Euclidean and  $p$ -norms of a vector  $x$ , respectively. The symbol  $x_i$  denotes the  $i$ th element of a vector  $x$ . Let  $\bar{\sigma}(M)$  and  $\|M\|_\infty$  denote the largest singular value and the induced  $\infty$ -norm of a matrix  $M$ . For Hermitian matrices  $M$ ,  $\bar{\lambda}(M)$  and  $\underline{\lambda}(M)$  denote the largest and smallest eigenvalues, respectively. The notation  $M > 0$  means that  $M$  is symmetric positive definite, and  $M^{\frac{1}{2}}$  denotes the unique positive definite square root of  $M > 0$ .

We consider linear systems:

$$\dot{x} = Ax + Bu + B_d d, \quad x(0) = x_0, \quad (1)$$

where  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $B_d \in R^{n \times p}$ , and  $x_0$  is a given initial state. The constraints for the state  $x(t) \in R^n$  and the control  $u(t) \in R^m$  are described as

$$x(t) \in X, \quad u(t) \in U, \quad \forall t \geq 0,$$

$$\begin{aligned} X &= \{x \in R^n : |x_i| \leq \gamma_i, \forall i\}, \\ U &= \{u \in R^m : |u_i| \leq \eta_i, \forall i\} \end{aligned} \quad (2)$$

with given constant vectors  $\gamma \in R^n$  and  $\eta \in R^m$ , while the disturbance  $d(t) \in R^p$ , which consists of a state dependent uncertainty  $d_1(t) \in R^{p_1}$  and a bounded disturbance  $d_2 \in R^{p_2}$ , satisfies

$$\begin{aligned} d(t) &:= \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \in D(x(t)), \quad \forall t \geq 0, \\ D(x) &:= D_1(x) \times D_2 \\ D_1(x) &:= \{d_1 \in R^{p_1} : \|d_1\| \leq \|x\|\} \\ D_2 &:= \{d_2 \in R^{p_2} : \|d_2\| \leq 1\}. \end{aligned} \quad (3)$$

Corresponding to  $d_1$  and  $d_2$ , matrices  $B_{d_1} \in R^{n \times p_1}$  and  $B_{d_2} \in R^{n \times p_2}$  are defined such that  $B_d = [B_{d_1}, B_{d_2}]$ .

We assume that a feedback gain  $K \in R^{m \times n}$  and a terminal set

$$X_f = \{x \in R^n : V(x) \leq \gamma_f\}$$

are given for  $V(x) := \sqrt{x^T P x}$  ( $P > 0$ ), such that the following conditions are satisfied.

**Assumption 2.1**  $X_f$  and  $K$  have the following relationship with constraint sets  $X$  and  $U$ :

$$\begin{aligned} X_f &\subset X \\ Kx &\in U, \quad \forall x \in X_f, \end{aligned}$$

**Assumption 2.2** For  $Q$  defined by

$$\begin{aligned} Q &:= -(PA_c + A_c^T P), \\ A_c &:= A + BK, \end{aligned} \quad (4)$$

it is satisfied that

$$\begin{aligned} 0 &< \sqrt{\bar{\lambda}(P)} \frac{\alpha_2}{\alpha_1} < \gamma_f, \\ \alpha_1 &:= \underline{\lambda}(Q) - 2\bar{\sigma}(PB_{d_1}) > 0 \\ \alpha_2 &:= 2\bar{\sigma}(PB_{d_2}). \end{aligned}$$

Assumption 2.1 states that the given feedback control  $u = Kx$  always satisfies the constraints in  $X_f$ , whereas Assumption 2.2 implies

$$\begin{aligned} \sup_{d \in D(x)} \dot{V}(x) &< 0, \quad \forall x \in R^n \setminus \Omega \\ \Omega &:= \left\{ x \in R^n : V(x) \leq \sqrt{\bar{\lambda}(P)} \frac{\alpha_2}{\alpha_1} \right\}. \end{aligned}$$

Therefore, these assumptions show that, by using the feedback control  $u = Kx$ , (i)  $X_f$  is a robustly controlled invariant set [14], and (ii)  $V(x)$  would be an RCLF [13], if the system were not constrained.

Under these assumptions, our goal is to construct a MPC method which guarantees the ultimate boundedness of the closed-loop without violating constraints for

any  $d(t) \in D(x(t))(t > 0)$ . Note that, while most of the existing methods consider the Property (i) above of the given feedback  $K$  only in the terminal set  $X_f$  for ensuring the stability of the closed-loop, we make use of the Property (ii) of  $K$  outside  $X_f$  as well as inside  $X_f$ .

### 3 Proposed MPC method

In most MPC methods, the following finite-horizon open-loop optimal control problem based on a plant nominal model:

#### (Nominal MPC)

$$\begin{aligned} & \min_{\hat{u}} J(x(t), \hat{u}(\cdot|t)) \\ & \text{s.t.} \\ & \hat{\dot{x}} = A\hat{x} + B\hat{u}, \quad \hat{x}(t|t) = x(t) \\ & \hat{u}(\tau|t) \in U, \quad \tau \in [t, t+T] \\ & \hat{x}(\tau|t) \in X, \quad \tau \in [t, t+T] \\ & \hat{x}(t+T|t) \in X_f \end{aligned} \quad (5)$$

is solved from the measured state  $x(t)$  at current time  $t$ , and the optimal control trajectory  $\hat{u}^*(\tau|t)$  is implemented until the new state measurement  $x(t+\delta)$  is obtained for the sampling time  $\delta$ . In (5), the terminal condition

$$\hat{x}(t+T|t) \in X_f$$

is introduced to ensure the stability of the closed-loop, and the terminal set  $X_f$  is chosen such that any states in  $X_f$  are steered to the origin without violating the constraints by a given feedback control law. (See *e.g.* [1] for the detail on how to choose the cost  $J$ , the terminal set  $X_f$ , and the local control law in  $X_f$  for nominal MPC.) Since this approach cannot explicitly incorporate model uncertainties, the optimization problem in (5) needs to be modified to handle the robust constrained problem described in Section 2.

One way to deal with the problem in Section 2 by MPC is to consider, instead of (5), the following min-max problem including a disturbance term  $\hat{d}$ . (See [1–10] for the detail of min-max MPC methods.)

#### (Min-max MPC)

$$\begin{aligned} & \min_{\hat{u}} \max_{\hat{d}} J(x(t), \hat{u}(\cdot|t), \hat{d}(\cdot|t)) \\ & \text{s.t.} \\ & \hat{\dot{x}} = A\hat{x} + B\hat{u} + B_d\hat{d}, \quad \hat{x}(t|t) = x(t) \\ & \left. \begin{aligned} & \hat{u}(\tau|t) \in U, \\ & \hat{x}(\tau|t) \in X, \\ & \hat{x}(t+T|t) \in X_f \end{aligned} \right\} \forall \hat{d}(\tau|t) \in D(\hat{x}(\tau|t)) \\ & \quad \quad \quad \tau \in [t, t+T]. \end{aligned} \quad (6)$$

One of the limitations of the min-max MPC approaches is that the optimization in (6) is too complex to be

solved online. Another difficulty is that the problem (6) is easily led to infeasibility especially for unstable systems, since it requires that a single open-loop control trajectory satisfies the given constraints for all possible disturbances.

In this paper, we construct a “nominal” MPC problem, which is able to handle the robust constrained problem in Section 2, by modifying the constraint sets  $U$ ,  $X$ ,  $X_f$  in (5) such that a control trajectory  $\hat{u}^*(\tau|t)$  is chosen from the feasible solutions of the min-max MPC in (6). To this end, we need to evaluate the worst-case prediction error of the nominal state  $\hat{x}$  in (5) using *a priori* information on model uncertainties.

In order to prevent the prediction error analysis for  $\hat{x}$  from becoming too conservative, we use the closed-loop prediction as follows:

$$\hat{u}(\tau|t) = K\hat{x}(\tau|t) + \tilde{u}(\tau|t) \quad (7)$$

where  $\tilde{u}$  is an open-loop trajectory and  $K$  is the feedback gain given in Section 2. By using (7), the nominal model in (5) and the real system in (1) are rewritten as

$$\hat{\dot{x}} = A_c\hat{x} + B\tilde{u}, \quad (8)$$

$$\dot{x} = A_c x + B\tilde{u} + B_d d. \quad (9)$$

The importance of the closed-loop prediction is also suggested by Bemporad [2, 12] for linear systems with bounded disturbances (*i.e.*  $B_{d1} = 0$  in (1)). However, in the case where  $B_{d1} \neq 0$ , since the disturbance set  $D$  depends on the state  $x$  of the real system as in (3), it is difficult to evaluate the prediction error from only the nominal model in (5) and the disturbance set in (3). In order to overcome this difficulty, we introduce the following scalar comparison system into the optimization problem in (5):

$$\begin{aligned} \dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, \quad w(t) = V(x(t)), \quad (10) \\ a_1 &= \frac{\alpha_1}{2\lambda(P)}, \quad a_2 = \frac{\alpha_2}{2\sqrt{\lambda(P)}}, \quad b_i = \|P^{\frac{1}{2}} B_i\|, \end{aligned}$$

where  $B_i$  is the  $i$ th column of  $B$ . The system in (10) is constructed based on the RCLF  $V(x)$  for the unconstrained system, and has the following property.

**Lemma 3.1** *For any  $\tilde{u}(\tau)$  ( $\tau \in [t, t+T]$ ), the states of the comparison system in (10) and the real system in (9) satisfy*

$$V(x(\tau)) \leq w(\tau), \quad \tau \in [t, t+T]. \quad (11)$$

In order to describe the modified constraint sets, we define the following trajectories given by  $A_c$ ,  $B_d$ ,  $K$ :

$$\begin{aligned} \zeta_1(t) &:= e^{A_c t} B_{d1}, \quad \zeta_2(t) := e^{A_c t} B_{d2}, \\ \xi_1(t) &:= K e^{A_c t} B_{d1}, \quad \xi_2(t) := K e^{A_c t} B_{d2}. \end{aligned}$$

The constraint sets  $U$ ,  $X$  are now modified as the following sets  $\widehat{U}$ ,  $\widehat{X}$  depending on  $w$ .

$$\begin{aligned}\widehat{X}(\tau, w(\cdot|t)) &= \{x \in R^n : |x_i| \leq \gamma_i - \widehat{\gamma}_i(\tau, w(\cdot|t)), \forall i\} \\ \widehat{U}(\tau, w(\cdot|t)) &= \{u \in R^m : |u_i| \leq \eta_i - \widehat{\eta}_i(\tau, w(\cdot|t)), \forall i\}\end{aligned}$$

where

$$\begin{aligned}\widehat{\gamma}_i(\tau, w(\cdot|t)) &= \int_0^{\tau-t} (\|\zeta_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\Delta(P)}} + \|\zeta_{2i}^T(s)\|_1) ds \\ \widehat{\eta}_i(\tau, w(\cdot|t)) &= \int_0^{\tau-t} (\|\xi_{1i}^T(s)\|_1 \frac{w(\tau-s|t)}{\sqrt{\Delta(P)}} + \|\xi_{2i}^T(s)\|_1) ds\end{aligned}$$

and  $\zeta_{1i}(s)$ ,  $\zeta_{2i}(s)$ ,  $\xi_{1i}(s)$ ,  $\xi_{2i}(s)$  denote the  $i$ th rows of  $\zeta_1(s)$ ,  $\zeta_2(s)$ ,  $\xi_1(s)$ ,  $\xi_2(s)$ , respectively. The constraints modified by  $\widehat{X}$ ,  $\widehat{U}$  above and the scalar system (10) have the following property.

**Theorem 3.1** For a given  $x(t) \in R^n$ , any open-loop trajectory  $\tilde{u}(\tau|t)$ ,  $\tau \in [t, t+T]$ , which satisfies

$$\begin{aligned}\dot{\hat{x}} &= A_c \hat{x} + B \tilde{u}, & \hat{x}(t|t) &= x(t) \\ \dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, & w(t|t) &= V(x(t)) \\ \hat{x}(\tau|t) &\in \widehat{X}(\tau, w(\cdot|t)), & \tau &\in [t, t+T] \\ \tilde{u}(\tau|t) + K \hat{x}(\tau|t) &\in \widehat{U}(\tau, w(\cdot|t)), & \tau &\in [t, t+T] \\ w(t+T|t) &\leq \gamma_f,\end{aligned}$$

also satisfies the constraints in (6) for all  $d(\tau) \in D(x(\tau))$ ,  $\tau \in [t, t+T]$ .

Based on Theorem 3.1, the optimization problem for the proposed MPC method is described for a given diagonal matrix  $R > 0$  as follows:

**(Proposed MPC)**

$$\begin{aligned}\min_u J(x(t), \tilde{u}(\cdot|t)) &:= \int_{\tau=t}^{t+T} \tilde{u}(\tau|t)^T R \tilde{u}(\tau|t) d\tau \\ \text{s.t.} & \\ \dot{\hat{x}} &= A_c \hat{x} + B \tilde{u}, & x(t|t) &= x(t) \\ \dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, & w(t|t) &= V(x(t)) \\ \tilde{u}(\tau|t) + K \hat{x}(\tau|t) &\in \widehat{U}(\tau, w(\cdot|t)), & \tau &\in [t, t+T] \\ \hat{x}(\tau|t) &\in \widehat{X}(\tau, w(\cdot|t)), \\ w(\tau|t) &\leq \omega, \\ w(t+T|t) &\leq \gamma_f.\end{aligned}\quad (12)$$

In (12), we introduce an additional constraint

$$w(\tau|t) \leq \omega, \quad \tau \in [t, t+T],$$

where  $\omega$  is a constant number satisfying  $\omega \geq \max\{V(x_0), \gamma_f\}$ . In Section 4, an additional assumption is imposed on  $\omega$  for ensuring the feasibility of the optimization problem at the “next” sampling time. Namely, although  $\omega$  is desired to be as large as possible for the feasibility at the current time, the value of  $\omega$  to guarantee the feasibility at the next time step is bounded, as described in Assumption 4.1.

Cost functions  $J$  of optimal control problems often play important roles in stability analysis of MPC methods. In this paper, we choose the cost function as in (12), which is probably the simplest one to ensure robust stability. Since the control problem based on this cost function minimizes the difference between the predicted control  $\tilde{u}(\tau|t)$  in (7) and the feedback control  $K \hat{x}(\tau|t)$ , it is necessary to choose  $K$  having a desirable performance at least for unconstrained systems by using robust control design methods [16, 17]. Robust analysis for other types of cost functions is one of the future issues.

**Remark 3.1** The second constraint in (12)

$$\dot{w} = -a_1 w + a_2 + \sum_{i=1}^m b_i |\tilde{u}_i|, \quad w(t|t) = V(x(t)) \quad (13)$$

is a nonlinear equation of  $\tilde{u}_i$ . By introducing a new variable  $v(\tau|t) \in R^m$  and using the property that  $R$  is diagonal, we modify the constraint in (13) to

$$\begin{aligned}\dot{w} &= -a_1 w + a_2 + \sum_{i=1}^m b_i v_i, & w(t|t) &= V(x(t)) \\ |\tilde{u}_i(\tau|t)| &\leq v_i(\tau|t), & i &= 1, \dots, m\end{aligned}$$

and the cost function  $J(x(t), \tilde{u}(\cdot|t))$  to  $J(x(t), v(\cdot|t))$ . Therefore, the modified problem has only linear constraints and gives the same solution as (12), since the optimal solution of the modified problem always satisfies  $|\tilde{u}_i(\tau|t)| = v_i(\tau|t)$ . The recast problem with the linear constraints and the quadratic cost function has free variables  $\tilde{u}_i(\tau|t)$  and  $v_i(\tau|t)$ .

**Remark 3.2** One way of implementing the proposed method is to discretize the optimal control problem in (12). That is, a discrete-time sequence

$$(\tilde{u}(t|t), \tilde{u}(t+\delta|t), \dots, \tilde{u}(t+\delta(N-1)|t)), \quad T = \delta N$$

is obtained by discretizing the state-space systems in (12), and the first element  $\tilde{u}(t|t)$  is applied until the next time step  $t+\delta$ . Another way is to apply the discrete-time method [18] for the discretized system of (1). In both cases, the discrete-time optimization can be reduced to a QP problem, since all constraints in (12) are linear as mentioned in Remark 3.1.

**Remark 3.3** In (12), the predicted nominal state  $\hat{x}$  typically converges to the terminal set faster than  $w$ , since  $w$  is the worst-case value of  $V(x)$  for all possible disturbances. Therefore, the computational burden can be decreased by introducing a control horizon  $T_u < T$  and by applying  $\tilde{u}$  and  $\omega$  of the following form:

$$\tilde{u}(\tau|t) = \begin{cases} \tilde{u}'(\tau|t), & \text{if } t \leq \tau \leq t+T_u \\ 0, & \text{if } \tau > t+T_u, \end{cases} \quad (14)$$

In this case,  $\omega$  in (12) can be modified as

$$\omega'(\tau) = \begin{cases} \omega, & \text{if } t \leq \tau \leq t+T_u \\ e^{-a_1(\tau-T_u)} \left( \omega - \frac{a_2}{a_1} \right) + \frac{a_2}{a_1}, & \text{if } \tau > t+T_u, \end{cases}$$

from (10) and  $\tilde{u}(\tau|t) = 0$  ( $\tau > t+T_u$ ).

## 4 Feasibility and stability results

As mentioned in the previous section, we need the following assumption prescribing upper bounds of  $\omega$  to ensure that the optimal control problem in (12) is feasible at each time.

**Assumption 4.1** The given  $\omega \geq \max\{V(x_0), \gamma_f\}$  in (12) satisfies

$$\begin{aligned} \omega \|\zeta_1\|_{\mathcal{L}_1(T)} &\leq \sqrt{\lambda(P)}(\min_i \gamma_i - \|\zeta_2\|_{\mathcal{L}_1(T)}) - \gamma_f \\ \omega \|\xi_1\|_{\mathcal{L}_1(T)} &\leq \sqrt{\lambda(P)}(\min_i \eta_i - \|\xi_2\|_{\mathcal{L}_1(T)}) \\ &\quad - \gamma_f \max_i \|K_i^T\|, \end{aligned}$$

where  $K_i$  denotes the  $i$ th row of  $K$  and

$$\|\zeta_1\|_{\mathcal{L}_1(T)} := \int_0^T \|\zeta_1(s)\|_{\infty} ds.$$

Note that Assumption 4.1 is given such that the same type of condition as Assumption 2.1 is satisfied for the modified constraint sets  $\widehat{X}$  and  $\widehat{U}$ . That is

$$X_f \subset \widehat{X}(\tau, w(\cdot|t)) \subseteq X \quad (15)$$

$$Kx \in \widehat{U}(\tau, w(\cdot|t)) \subseteq U, \quad \forall x \in X_f. \quad (16)$$

Therefore, Assumption 4.1 is a sufficient condition for Assumption 2.1. It is also important to note that, in the case where Assumption 4.1 cannot be satisfied for any  $\omega > \max\{V(x_0), \gamma_f\}$ , we need to consider a smaller terminal set  $X_f$  or to modify the feedback gain  $K$  to satisfy Assumption 4.1.

The following theorem describes the properties of the feasibility and the ultimate boundedness of the proposed MPC method.

**Theorem 4.1** Assume the optimization in (12) is feasible at  $t = 0$  for  $\omega$  which satisfies Assumption 4.1. Then, the proposed MPC method has the following properties:

- (i) the optimization in (12) is feasible at each  $t > 0$ ,
- (ii) for any  $\mu > a_2/a_1$ , there exists  $t_c$  such that

$$\|x\| \leq \frac{\mu}{\sqrt{\lambda(P)}}, \quad \forall t \geq t_c.$$

Theorem 4.1 tells us that, if the MPC problem is feasible at  $t = 0$ , the state  $x(t)$  converges into a ball around the origin with radius of  $a_2/a_1 \sqrt{\lambda(P)}$ . Particularly, if there does not exist a non-vanishing uncertain term  $d_2$ , the state is steered to the origin. It is also important to notice that the control law of the proposed method converges to the given feedback law  $u = Kx$ . Once the state is steered into the robustly controlled invariant set  $X_f$ , the control law is completely switched to the feedback law  $u = Kx$ , since it is the optimal control in  $X_f$  in terms of the cost function in (12).

## 5 Numerical Example

Consider the following uncertain system:

$$\dot{x} = \begin{bmatrix} 0 & 1 + \alpha \\ 1 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \beta, \quad (17)$$

where the state and control constraints are given as

$$\begin{aligned} |x_i(t)| &\leq 1, \quad i = 1, 2 \\ |u(t)| &\leq 2, \end{aligned}$$

and the bounds of the uncertain parameters

$$|\alpha(t)| \leq 0.1, \quad \|\beta(t)\| \leq 0.1,$$

are given as *a priori* information. The uncertain system in (17) is described as the form in (1) with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ B_{d_1} &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad B_{d_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \end{aligned}$$

A feedback gain  $K$ , a Lyapunov function  $V(x)$  and a terminal set  $X_f$ , which satisfy Assumption 2.1 and 2.2, are chosen as

$$\begin{aligned} K &= [-1.66, -2.32], \quad V(x) = \|x\|, \quad (18) \\ X_f &= \{x \in R^n : V(x) \leq 0.56\} \end{aligned}$$

In this case, the comparison system (10) is obtained as

$$\dot{w} = -0.647w + 0.1 + 1.41|\tilde{u}|. \quad (19)$$

We choose the horizons as  $T_u = 1.0$ ,  $T = 3.0$ [sec] and the upper bound of  $w$  as  $\omega = 2$ , such that Assumption 4.1 is satisfied.

The proposed method is now applied to the systems in (17) and (19) discretized with sampling time 0.1[sec]. We show the simulation result for unknown parameters  $\alpha = -0.1$  and  $\beta(t)$  chosen as a step signal changing its value randomly at every 0.5[sec]. The trajectory for 10[sec] of the state starting from  $x_0 = [-1, -1]$  is shown by the solid line in Figure 1. The dashed and dotted lines show the terminal set and the ultimate bound respectively. From Figure 1, it can be seen that the trajectory of the state goes into the terminal set and achieves ultimate boundedness against the uncertainties. Figure 2 shows with the solid line the control trajectory obtained by the proposed method. The dashed line shows the control trajectory by the given feedback controller in (18). As shown in Figure 2, the given feedback controller violates the constraint, since it is designed without taking account of the constraints. On the other hand, the solid line in Figure 2 shows that the control obtained by the proposed MPC satisfies the given constraint.

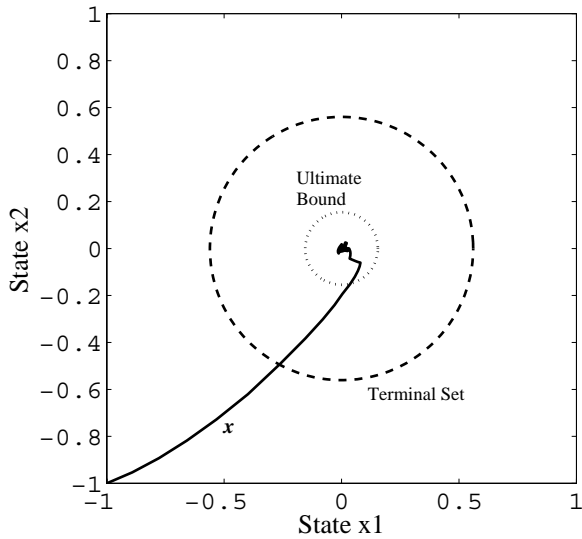


Figure 1: Trajectory of the state

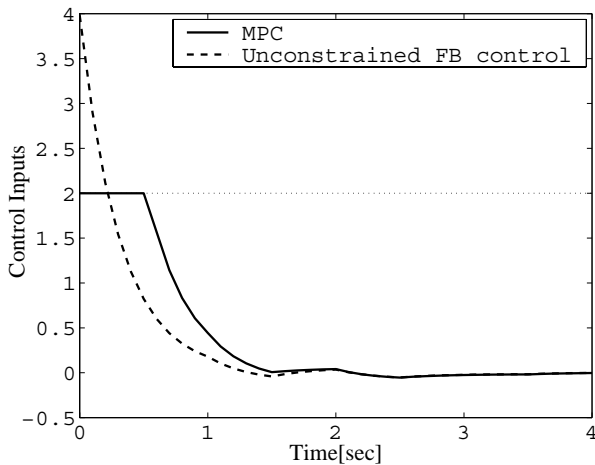


Figure 2: Time plots of the controls

## 6 Conclusion

In this paper, we have proposed a new robust MPC method for constrained linear uncertain systems. The merits of the proposed method are summarized as (i) the control optimization is reduced to a QP rather than a min-max problem, (ii) state dependent uncertainties can be handled as well as bounded disturbances, and (iii) the robust stability of the closed-loop can be ensured by a terminal condition introduced in this paper. In order to obtain these properties, we have introduced an additional comparison model for worst-case analysis based on an RCLF for the unconstrained system. By using the comparison model, we have transformed the given robust MPC problem to a nominal one without uncertain terms. We have also shown that the terminal condition based on the comparison model ensures the feasibility and stability of the proposed method. Moreover, it has been shown that, since the terminal condition is described as linear constraints, the control optimization can be reduced to a QP problem.

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