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MODEL PREDICTIVE CONTROL AND STATE ESTIMATION: A NETWORK EXAMPLE

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Abstract: Model Predictive Control (MPC) is of interest because it is one of the few control design methods which preserves standard design variables and yet handles constraints. This paper focuses on a stochastic MPC problem with constraints specified in a probabilistic sense. Our aim is to study the incorporation of state estimates into the MPC problem. The original problem can be approximated by a deterministic constrained MPC problem for the conditional mean by absorbing the state estimates' covariances into the constraints. Our idea is explored in a standard discrete time Linear Quadratic Gaussian problem, and is demonstrated with a simple application in network congestion control.

Keywords: predictive control,constraints,state estimation,communication networks,traffic control.

1. INTRODUCTION

Model Predictive Control (MPC) or Receding Horizon Control is a method in which the current control action is obtained online by solving a finite horizon open-loop optimal control problem from the current system state or from its estimate based on output measurements. It yields a finite length open loop control sequence from the current state estimate. The first element of this sequence is applied to the plant. Then the output response is measured, the state estimate updated and the optimization re-solved at the next time instant, thereby producing an implicit feedback control via the state estimator. An important feature of MPC is the capacity to handle constraints in the open loop optimization, see Mayne *et al.* (2000). These constraints could be on the inputs/controls, system outputs or states.

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The bulk of MPC analysis in the constrained case has focused on the establishment of feasibility and stability with full state feedback control. Few people have studied the interaction between the constraints and state estimation errors when state estimation is (necessarily) included into the picture. This is to be considered in a constrained linear quadratic gaussian (LQG) problem. Because of the existence of noise in the system, it is necessary to pose the constrained optimization problem in a stochastic form. Our state estimates are provided by a Kalman Filter, which yields an estimate of the plant state and a state estimate error covariance matrix. The core issue is that in a constrained optimization problem, these state estimation errors could lead to violation of the constraints at future time. A simple method is proposed for incorporating the state estimates and covariance information into the constraints to yield a deterministic problem. In Section 2, the problem is formulated for a linear system with quadratic criterion and gaussian noise term and initial state. In Section 3, the idea is demonstrated in a communication network congestion control problem developed by Altman *et al.* (1999) in which explicit constraints are introduced.

2. PROBLEM FORMULATION

Here MPC is considered to be applied to a k dimensional linear discrete time system with quadratic criterion, gaussian initial state and noise term:

$$\mathbf{x_{n+1}} = \mathbf{A}\mathbf{x_n} + \mathbf{B}\mathbf{u_n} + \mathbf{G}\phi_n, \qquad (1)$$
$$\mathbf{x_n} = (x_n^1, \ldots, x_n^k)^T$$

where $\phi_n \sim N(0, \sigma_n^2), \mathbf{x_0} \sim N(\bar{\mathbf{x}}_\mathbf{0}, \mathbf{X_0})$. Due to the underlying stochastic nature problem, the criterion to be minimized is an expectation:

$$J(N, \mathbf{x_n}) = E\Big[\mathbf{x^T_{n+N}}\mathbf{P_N}\mathbf{x_{n+N}} + \sum_{i=0}^{N-1}\left(\mathbf{x^T_{n+i}}\mathbf{Q_i}\mathbf{x_{n+i}} + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right)\Big]. \qquad (2)$$

It is supposed that a set of "soft" probabilistic constraints is required to be satisfied as follows:

$$P(x^j_{n+i} \leq \beta^j_i) \geq \bar{p}^j_i, i = 0, 1 \ldots N, j = 1, \ldots k. \qquad (3)$$

where x^j_{n+i} is the jth element of $\mathbf{x_{n+i}}$. These probabilistic constraints reflect the infeasibility of satisfying almost surely hard limits with gaussian processes while capturing the need to avoid too frequent excess values. In queuing problems, where overflowing queues lead to retransmission which can be tolerated provided sufficiently infrequent, this style of constraint makes sense. However, (3) leads the problem into a stochastic programming task which is difficult to solve. The problem will be converted into a deterministic form which could be solved by quadratic programming.

As the system is linear with gaussian initial state and process noise, in the unconstrained case, all $\{\mathbf{x_n}\}$ have joint normal distributions $N(\bar{\mathbf{x}}_\mathbf{n}, \mathbf{X_n})$. However, this may not be true with the presence of constraints, in which case the resultant controller can be nonlinear. Here for ease we make this 'Normal' assumption. Later in Section 3, it will be shown that this is practically a conservative assumption. The conditional mean of $\mathbf{x_{n+i}}$ given data up to time n, $\bar{\mathbf{x}}_\mathbf{n+1|n}$, and its covariance $\mathbf{X_{n+i|n}}$, evolve into the future as

$$\bar{\mathbf{x}}_\mathbf{n+i|n} = \mathbf{A}\bar{\mathbf{x}}_\mathbf{n+i-1|n} + \mathbf{B}\mathbf{u_{n+i-1}} \qquad (4)$$
$$\mathbf{X_{n+i|n}} = \mathbf{A}\mathbf{X_{n+i-1|n}}\mathbf{A^T} + \mathbf{G}\sigma_n^2\mathbf{G^T} \qquad (5)$$

and $(x^j_{n+i} - \bar{x}^j_{n+i|n})/(X^{jj}_{n+i|n})^{\frac{1}{2}}$ has standard normal distribution. In (3), for every given $\bar{p}^j_i$ a corresponding β^{j*}_i could be found in the Standard Normal Distribution Table such that $\Phi(\beta^{j*}_i) = \bar{p}^j_i$. $\Phi(\cdot)$ is the Standard Normal Distribution function. As (3) is equivalent to $P\left(\frac{x^j_{n+i} - \bar{x}^j_{n+i|n}}{(X^{jj}_{n+i|n})^{\frac{1}{2}}} \leq \frac{\beta^j_i - \bar{x}^j_{n+i|n}}{(X^{jj}_{n+i|n})^{\frac{1}{2}}}\right) \geq \bar{p}^j_i$, the constraint can be recast as:

$$\frac{\beta^j_i - \bar{x}^j_{n+i|n}}{(X^{jj}_{n+i|n})^{\frac{1}{2}}} \geq \beta^{j*}_i \Leftrightarrow \bar{x}^j_{n+i|n} \leq \beta^j_i - \beta^{j*}_i (X^{jj}_{n+i|n})^{\frac{1}{2}},$$
$$(i = 0, 1 \ldots N, j = 1, \ldots, k). \qquad (6)$$

Thus the probabilistic constraint may be replaced by a deterministic one on the conditional mean process of the state. The covariance of the state estimate evolves as above in (5), which has the effect of modifying the effective constraints. Typically this covariance increases with i and in (3) this leads to successively more stringent conditions being applied on the control.

Next we represent the criterion (2) in terms of $\bar{\mathbf{x}}_\mathbf{n+i|n}$ and $\mathbf{u_{n+i}}$:

$$\begin{aligned}
&J(N, \mathbf{x_n}) \\
&= E\left[tr(\mathbf{x^T_{n+N}}\mathbf{P_N}\mathbf{x_{n+N}})\right] \\
&\quad + \sum_{i=0}^{N-1}\left\{E\left[tr(\mathbf{x^T_{n+i}}\mathbf{Q_i}\mathbf{x_{n+i}})\right] + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right\} \\
&= tr\left[E(\mathbf{x_{n+N}}\mathbf{x^T_{n+N}})\mathbf{P_N}\right] \\
&\quad + \sum_{i=0}^{N-1}\left\{tr\left[E(\mathbf{x_{n+i}}\mathbf{x^T_{n+i}})\mathbf{Q_i}\right] + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right\} \\
&= tr\left[(\mathbf{X_{n+N|n}} + \bar{\mathbf{x}}_\mathbf{n+N|n}\bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+N|n})\mathbf{P_N}\right] \\
&\quad + \sum_{i=0}^{N-1}\left\{tr\left[(\mathbf{X_{n+i|n}} + \bar{\mathbf{x}}_\mathbf{n+i|n}\bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+i|n})\mathbf{Q_i}\right]\right. \\
&\quad \left. + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right\} \\
&= tr(\mathbf{X_{n+N|n}}\mathbf{P_N}) + \sum_{i=0}^{N-1} tr(\mathbf{X_{n+i|n}}\mathbf{Q_i}) \\
&\quad + \bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+N|n}\mathbf{P_N}\bar{\mathbf{x}}_\mathbf{n+N|n} + \sum_{i=0}^{N-1}\left(\bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+i|n}\mathbf{Q_i}\bar{\mathbf{x}}_\mathbf{n+i|n}\right. \\
&\quad \left. + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right) \qquad (7)
\end{aligned}$$

From (5) the evolution of state covariance $\mathbf{X_{n+i|n}}$ will not be changed by the controls. Therefore, the optimal control problem, which is to minimize (2) for the system of (1) subject to the constraints given by (3), is approximated by a new problem, which is to minimize

$$J'(N, \bar{\mathbf{x}}_\mathbf{n|n}) = \bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+N|n}\mathbf{P_N}\bar{\mathbf{x}}_\mathbf{n+N|n} + \sum_{i=0}^{N-1}\left(\bar{\mathbf{x}}^\mathbf{T}_\mathbf{n+i|n}\mathbf{Q_i}\bar{\mathbf{x}}_\mathbf{n+i|n} + \mathbf{u^T_{n+i}}\mathbf{R_i}\mathbf{u_{n+i}}\right) \qquad (8)$$

for the system of (4) subject to the constraints given by (5) (6). The new problem is about steering the mean in a deterministic constrained form, which could be solved by Quadratic Programming.

To apply MPC a Kalman Filter(KF) is used as the state estimator. Then MPC is modified as follows:

(1) Time n, obtain $\mathbf{\hat{x}_{n|n}}$ and $\mathbf{X_{n|n}}$ from KF, compute future covariances $\mathbf{X_{n+1|n}} \dots \mathbf{X_{n+N|n}}$ from (5).

(2) Modify the constraints into the form (6).

(3) Solve the deterministic problem with initial state $\mathbf{x_n} = \mathbf{\hat{x}_{n|n}}$ to derive the optimal sequence of controls $\{\mathbf{u_n}, \dots, \mathbf{u_{n+N}}\}$.

(4) Apply $\mathbf{u_n}$ into the system.

(5) Timer+1, repeat from Step 1.

This is not certainty equivalence control, since the covariance information is included in the control action computation via the constraints. Certainty equivalence approaches rely on substituting the best available state estimate for the state in a full state feedback controller — the covariance does not enter the picture. Because of the constraint in MPC, the covariance information becomes important for the avoidance of frequent breaches of bounds. This introduces a further degree of caution into the control design.

In the next section an illustrative example is given in communication network congestion control with delays. It is a special case in that the state is known exactly at time n, so that $\mathbf{X_{n|n}} = 0$. The KF recursion (4–5) produces the downstream means and covariances. They are used in the constraints.

3. A SIMPLE EXAMPLE

In this section, the control of Available Bit Rate (ABR) traffic in an Asynchronous Transfer Mode (ATM) telecommunications network is considered. This is a multi-source, single-buffer congestion control problem.

3.1 *ABR Congestion Control*

The ABR service plays a central role in regulating the network traffic. The current standards for ATM traffic management are built on the foundation of a rate based (rate matching) flow control scheme, see details in Imer *et al.* (2001). ABR traffic has a variable available total rate determined by the higher priorities CBR, VBR traffic prevailing. The goal of ABR service congestion control is to provide fairness among all links with a minimal cell loss ratio and maximal utilization of network sources. The control challenge is to regulate ABR source rates to utilize maximally the available capacity as it varies while respecting the requirement not to overflow buffer node queues too frequently. There is a fundamental difficulty in obtaining good congestion control performance, namely the presence of action delay, which causes the current effective source rates to be decided by the bottleneck several steps ago. To deal with delay, Mascolo (1999) studied this problem using classical control techniques, Altman *et al.* (1999) viewed congestion control as a stochastic problem with action delay and developed a class of certainty equivalent controllers, which is suboptimal, Imer and Başar (1999) viewed the problem as a team problem with information delays and formulated it into a standard discrete time LQG problem in an extended state space. Also Imer *et al.* (2001) summarized a robust adaptive controller under uncertainty to delays.

The framework adopted here is that a number of sources share an intermediate bottleneck node and that the bottleneck node uses both available service and queue length to compute the data transmitting rates for the ABR sources. The modeling framework and criterion are basically the same as in Altman *et al.* (1999), explicitly taking delay into account, and treating the available bit rate service as an autoregressive(AR) process driven by white noise. The action delays and the AR process are incorporated into the states by augmenting the system dimension. Consequently, the system model is posed in the form of (1). Differently from Altman *et al.* (1999) and Imer and Başar (1999), in which an optimal controller is derived in a delay-free situation and then the delay is incorporated without a constraint, this paper introduces an explicit constraint on the appropriately shifted queue length in the probabilistic sense presented earlier. Then it is possible to apply the approach proposed in Section 2 to obtain an open-loop optimal controller which is not certainty equivalent.

3.2 *Problem Formulation*

Let; q_n be the queue length at the node, r_{mn} be the effective data rate of the mth source, and μ_n be the effective service rate available for ABR traffic at the node at the beginning of the nth time slot. The dynamics of the queue length evolve according to:

$$q_{n+1} = q_n + \sum_{m=1}^{M} r_{mn} - \mu_n, \tag{9}$$

$$\mu_n = \mu + \xi_n, \tag{10}$$

$$\xi_n = \sum_{i=1}^{p} \alpha_i \xi_{n-i} + \phi_{n-1}, \tag{11}$$

where μ is the known constant nominal service rate, $\{\xi_n\}$ is a pth order AR process driven by $\{\phi_n\}$, a zero-mean *i.i.d.* gaussian sequence with finite variance σ_n^2, and $\alpha_i, i = 1, \ldots, p$ are known coefficients. The effective source rate r_{mn} is a delayed action which was decided by the node several steps ago. Let d_m denote this action delay and $v_{m,n-d_m}$ denote the command decision for source m at time $n - d_m$. Hence $v_{m,n-d_m} = r_{mn}$. The objective function to be minimized is given by:

$$\begin{aligned} J =& \frac{1}{N} E\Big\{ \sum_{i=1}^{N} \big[(q_{n+i} - \bar{Q})^2 \\ &+ \sum_{m=1}^{M} \frac{1}{c_m^2} (r_{m,n+i} - a_m \mu_{n+i})^2\big]\Big\}, \end{aligned} \tag{12}$$

where; $\bar{Q}$ is the target queue length, c_ms are some positive constants which serve as weighting terms to prioritize relative importance among different sources, a_ms are fairness indices that serve to share the available bandwidth among the sources. If a fair sharing is desired, a_ms would be chosen as $a_1 = a_2 = \cdots = a_M = 1/M$. The first additive term represents a penalty for deviation from a desirable queue length, the second additive term is a measure of quality with which the input rate for each source tracks a given fraction of the ABR service.

By changing variables:

$$x_n^q := q_n - \bar{Q}, \quad u_{mn} := v_{mn} - a_m \mu, \tag{13}$$

which will serve as our state and control respectively, the dynamics of queue length can be re-written as:

$$x_{n+1}^q = x_n^q + \sum_{m=1}^{M} u_{m,n-d_m} - \xi_n, \tag{14}$$

$$\xi_{n+1} = \sum_{i=1}^{p} \alpha_i \xi_{n+1-i} + \phi_n,$$

and the objective function J can be re-written as

$$\begin{aligned} J =& \frac{1}{N} E\Big\{ \sum_{i=1}^{N} \big[{x_{n+i}^q}^2 \\ &+ \sum_{m=1}^{M} \frac{1}{c_m^2} (u_{m,n+i-d_m} - a_m \xi_{n+i})^2\big]\Big\}. \end{aligned} \tag{15}$$

To decide the controls by applying the approach posed in Section 2, the system should be presented in the form of (1). This is realized by augmenting the system as follows:

- Write the following d_i dimensional systems to accommodate the mismatched time indices in (14) and (15),

$$\mathbf{x^s_{i,n+1}} = \mathbf{A^s_i} \mathbf{x^s_{i,n}} + \mathbf{B^s_i} u^s_{i,n}, \tag{16}$$

$$y^s_{i,n} = \mathbf{C^s_i} \mathbf{x^s_{i,n}} \tag{17}$$

$$\mathbf{A^s_i} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{d_i}, \mathbf{B^s_i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{C^s_i} = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}, \quad (i = 1, \ldots, M).$$

- Use the following p-dimensional system to represent the AR process.

$$\mathbf{x^{AR}_{n+1}} = \mathbf{A^{AR}} \mathbf{x^{AR}_n} + \mathbf{B^{AR}} \phi_\mathbf{n} \tag{18}$$

$$y^{AR}_n = \mathbf{C^{AR}} \mathbf{x^{AR}_n} \tag{19}$$

$$\mathbf{A^{AR}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \alpha_p & \alpha_{p-1} & \alpha_{p-2} & \ldots & \alpha_1 \end{pmatrix}_p,$$

$$\mathbf{B^{AR}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{C^{AR}} = (0 \ \ldots \ 0 \ 1).$$

- Rewrite the node system (14) as,

$$x^q_{n+1} = x^q_n + \sum_{i=1}^{M} y^s_{i,n} - y^{AR}_n. \tag{20}$$

- Incorporate these three systems together into an augmented system accounting for the action delays internally.

$$\mathbf{x_{n+1}} = \mathbf{A}\mathbf{x_n} + \mathbf{B}\mathbf{u_n} + \mathbf{G}\phi_n, \tag{21}$$

$$\mathbf{x_n} = \begin{pmatrix} \mathbf{x^s_n} \\ x^q_n \\ \mathbf{x^{AR}_n} \end{pmatrix}, \mathbf{u_n} = \begin{pmatrix} u^s_{1,n} \\ \vdots \\ u^s_{n,n} \end{pmatrix},$$

$$
\mathbf{A}=\begin{pmatrix}
\mathbf{A_1^s} & & & 0 & 0 & \dots & 0\\
& \ddots & & \vdots & \vdots & & \vdots\\
& & \mathbf{A_M^s} & 0 & 0 & \dots & 0\\
\mathbf{C_1^s} & \dots & \mathbf{C_M^s} & 1 & & -\mathbf{C^{AR}} & \\
0 & \dots & 0 & 0 & & & \\
\vdots & & \vdots & \vdots & & \mathbf{A^{AR}} & \\
0 & \dots & 0 & 0 & & &
\end{pmatrix},
$$

$$
\mathbf{B}=\begin{pmatrix}
\mathbf{B_1^s} & & \\
& \ddots & \\
& & \mathbf{B_M^s}\\
0 & \dots & 0\\
0 & \dots & 0\\
\vdots & & \vdots\\
0 & \dots & 0
\end{pmatrix},\mathbf{G}=\begin{pmatrix}
0\\ \vdots\\ 0\\ 0\\ \mathbf{B_{AR}}
\end{pmatrix}.
$$

Next a set of probabilistic constraints, which is related to the upper bound on the queue length (or the shifted queue length x_n^q), will be introduced into our problem,

$$
P(x_{n+i}^q < \beta_i) > p_i,\ \ i = d_1+1, 1, \dots, N. \quad (22)
$$

This constraint says that to some small extent losing and re-transmitting a certain proportion of data is tolerable — p_i should be chosen close to 1. d_1 denotes the smallest delay among the sources. Due to the time delay, at time n states $x_n^q \cdots x_{n+d_1}^q$ are not controllable, therefore, they should be unconstrained. $x_{n+d_1+1}^q$ is the first controllable state and then only through u_n^1.

Now apply the method developed in Section 2 by modeling x_0^q and $\{\xi_n\}$ as gaussian. The state is assumed to remain gaussian at all subsequent times. The mean $\mathbf{\bar{x}_{n+i}}$ and covariance $\mathbf{X_{n+i}}$ of the state evolve as in (4) and (5). As x_{n+i}^q is normally distributed $x_{n+i}^q \sim N(\bar{x}_{n+i}, X_{n+i})$. Hence, as shown in Section 2, the constraint (22) may be replaced by

$$
\frac{\beta_i - \bar{x}_{n+i}^q}{{X_{n+i}^q}^{\frac{1}{2}}} \geq \beta_i^* \quad \Leftrightarrow \quad \bar{x}_{n+i}^q \leq \beta_i - \beta_i^* {X_{n+i}^q}^{\frac{1}{2}},
$$

where $\Phi(\beta_i^*) = p_i$, $\Phi(\cdot)$ is the Standard Normal Distribution function. As is mentioned in Section 2, the open-loop covariance X_{n+i}^q is typically increasing. For large N, this will lead to very tight constraints at the end of the horizon and a feasible solution may not exist. In practice, MPC strategy applies the first control element to the system based on full state information at each step, which is a closed-loop control scenario. Therefore the closed-loop state covariance should be included to modify the constraints instead of the pessimistic open-loop covariance. The first controllable x^q at time n is $x_{n+d_1+1}^q$, then $X_{n+d_1+1}^q$ should be act as closed-loop covariance. Also non-negative source rates and non-negative queue length should be considered in the constraints. Finally, the constraint set becomes

$$
\begin{aligned}
&\bar{x}_{n+i}^q \leq \beta_i - \beta_i^* {X^q}_{n+d_1+1}^{\frac{1}{2}}, i = d_1+1 \dots N\\
&u_{n+i}^j \geq -a_m\mu\\
&x_{n+i}^q \geq -\bar{Q}, \quad i = 1 \dots N
\end{aligned} \quad (23)
$$

Next the criterion will be represented in terms of $\mathbf{\bar{x}_n}$ and $\mathbf{u_n}$. Notice that:

$$
x_n^q = \mathbf{C_q x_n}, \mathbf{C_q} = (\overbrace{0,\dots,0}^{\sum_{i=1}^M d_i}, 1, 0, \dots, 0)
$$

and

$$
a_m\xi_n = \mathbf{C_{a_m\xi} x_n}, \mathbf{C_{a_m\xi}} = (\overbrace{0,\dots,0}^{1+\sum_{i=1}^M d_i}, a_m, 0, \dots, 0),
$$

(m=1 ... M).

Let $\mathbf{R} = \begin{pmatrix} \frac{1}{c_1^2} & & \\ & \ddots & \\ & & \frac{1}{c_M^2}\end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_M\end{pmatrix}$ and $\mathbf{C_{a\xi}} = \begin{pmatrix}\mathbf{C_{a_1\xi}}\\ \vdots \\ \mathbf{C_{a_M\xi}}\end{pmatrix}$.

Therefore,

$$
\begin{aligned}
J &= \frac{1}{N}E\Big\{\sum_{i=1}^N \big[(x_{n+i}^q)^2\\
&+ (\mathbf{u_{n+i}} - \mathbf{a}\xi_{n+i})^T\mathbf{R}(\mathbf{u_{n+i}} - \mathbf{a}\xi_{n+i})\big]\Big\}\\
&= \frac{1}{N}\sum_{i=1}^N \Big\{E\left[\mathbf{C_q x_{n+i} x_{n+i}^T C_q^T}\right]\\
&+ E\left[(\mathbf{u_{n+i}} - \mathbf{C_{a\xi}x_{n+i}})^\mathbf{T}\mathbf{R}(\mathbf{u_{n+i}} - \mathbf{C_{a\xi}x_{n+i}})\right]\Big\} \quad (24)\\
&= \frac{1}{N}\sum_{i=1}^N \Big[\mathbf{C_q} tr(\mathbf{X_{n+i}})\mathbf{C_q^T}\\
&+ \mathbf{\bar{x}_{n+i}^T}(\mathbf{C_q^T C_q} + \mathbf{C_{a\xi}^T R C_{a\xi}})\mathbf{\bar{x}_{n+i}} - \mathbf{\bar{x}_{n+i}^T C_{a\xi}^T R u_{n+i}}\\
&\quad - \mathbf{u_{n+i}^T R C_{a\xi} \bar{x}_{n+i}} + \mathbf{u_{n+i}^T R u_{n+i}}\Big].
\end{aligned}
$$

Since the evolution of the state covariance $\mathbf{X_n}$ is uncorrelated with the controls $\mathbf{u_n}$, minimizing J is equivalent to minimizing

$$
\begin{aligned}
J' = \frac{1}{N}\sum_{i=1}^N &\Big[\mathbf{\bar{x}_{n+i}^T}(\mathbf{C_q^T C_q} + \mathbf{C_{a\xi}^T R C_{a\xi}})\mathbf{\bar{x}_{n+i}} \quad (25)\\
&- \mathbf{\bar{x}_{n+i}^T C_{a\xi}^T R u_{n+i}} - \mathbf{u_{n+i}^T R C_{a\xi}\bar{x}_{n+i}} + \mathbf{u_{n+i}^T R u_{n+i}}\Big]
\end{aligned}
$$

Minimizing (25) subject to (4-5) and(23), the matrixes $\mathbf{A}, \mathbf{B}, \mathbf{G}$ are given by (21), is our new finite horizon problem to be solved at each time. This new problem differs from certainty equivalence by seeking a new balance between the performance requirement and the constraints.

3.3 *Simulation Results*

The simulation is of a single congested node accessed by three sources with delays $d_1 = 2, d_2 = 4, d_3 = 6$. The fairness indices are taken to be $a_1 = a_2 = a_3 = 1/3$. The AR process is assumed to be 2nd order with parameters $\alpha_1 = 0.4, \alpha_2 = 0.4$. The finite horizon should be taken bigger than

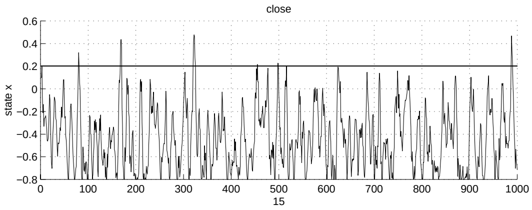

Fig. 1. State x_n

the biggest action delay, so $N = 10$. Assume the relative full queue length is 1, the initial relative queue length is 0.9 and the nominal queue length is 0. The probabilistic constraint is $P(x_{n+i} \leq 0.8) \geq 0.9505$, the corresponding $\beta^* = 1.65$. The total simulation time is 1000.

Figure 1 is the plot of the state x_n, the queue length deviation from the nominal value $\bar{Q}$. The values beyond 0.2 represent that the queue is overflowing. In this simulation result, there are 15 steps in 1000 overflowed, which is a smaller proportion than our expected 5%. The reason is that: with the 'Normal' assumption of states, the constraints are modified to obtain a control that moves the state mean appropriately to achieve the 5% performance while keeping the normal shape of the probability density function(PDF) of the queue. However, the queue can never go below 0 and an upper bound is also in the picture; hence the PDF has a cutoff on the left and is skewed to the left near the upper bound. Because the shape of PDF is not normal, the queue has less overflowing chance than the expected value. The 'Normal' assumption is actually a conservative assumption. Figure 2 shows how well u_n^ms track a given fraction of ξ_n. The source rates controllers u_n^1, u_n^2, u_n^3 have means $0.0043, -0.0015, -0.0015$ and covariances $0.0125, 0.0044, 0.0043$ separately. The controller tends to slow down those source rates having bigger delays and use the less delayed source rates to keep the queue length around a desired value. This results in an unfair distribution of ABR among sources. To improve fairness for sources with larger delays, one way is to add a bigger penalty to these source rate deviations from the given fraction of service rate, namely play with $c_1,\ c_2,\ c_3$. The other way is to include l_2 norms of the difference between any two source rates in the criterion.

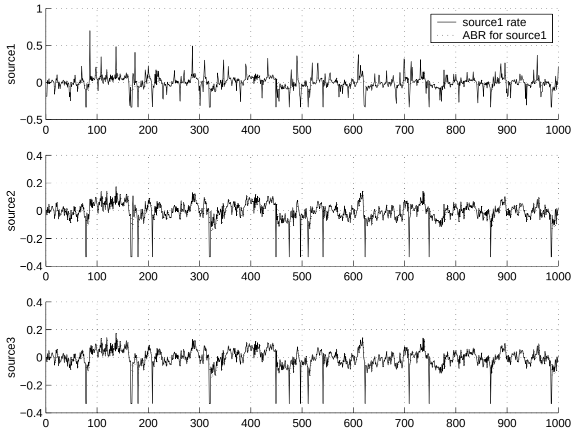

Fig. 2. Source rate deviations from nominal value($a_m\mu$) u_n^m (solid lines) and $a_m\xi_n$(dashed lines)

4. CONCLUSION

This paper has presented an introduction to the idea of incorporating state estimates and their covariances into MPC and analysis of their interaction. This idea was demonstrated by an application of MPC to a constrained stochastic network control problem.

Since the problem is linear quadratic and with linear constraints, an explicit continuous piecewise linear controller is obtained. This will give a better understanding of the PDF of the queue length. And fairness should be a issue in the future study.

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