Experimental Certification of Jet Engine Controllers: 
*Generalized Stability Margin Inference for a Large Number of MIMO Controllers*

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Motivation

- The demand for efficiency and performance pushes engine operation to its physical limits (Many constraints to meet to avoid rapid degradation and operational instabilities)

Joint Strike Fighter (Short Take Off/Vertical Landing)

The maximum efficiency for a compressor and fan is attained at near stall and surge margin. (may cause engine operation instability)
Outline and Background

I. Controller Certification
   - Single uncertain MIMO engine with a fixed model
   - A large number of possible constraints
     - must be satisfied, but a good design method
     - 20 constraints and up to 4 of constraints are active at any one instance, 6196 possible combinations

II. Less Conservative Margin and $\nu$-gap Computations

III. Experiments for Estimation of the Margin with an error bound
Outline and Background

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III. Experiments for Estimation of the Margin with an error bound
Controller Certification

- Looking for the best tasting wines by sampling ...
- I cannot test all the wines here
  - although I would like to
- Which wines should I test?
  - ideas of closeness - metric distance
  - informative choices for testing
  - quality measures and inference for neighbors

The Main Tools

- **Generalized Stability Margin** $b_{P,C}$

\[
T(P, C) = \begin{pmatrix}
P(I + CP)^{-1}C & P(I + CP)^{-1} \\
(I + CP)^{-1}C & (I + CP)^{-1}
\end{pmatrix}
\]

\[
b_{P,C} = \begin{cases}
    (\|T(P, C)\|_\infty)^{-1} & \text{if } T(P, C) \text{ is stable,} \\
    0 & \text{else.}
\end{cases}
\]

\[
b_{P,C,\omega} = \begin{cases}
    (\sigma_{\max} [T(P, C, \omega)])^{-1} & \text{if } T(P, C) \text{ is stable,} \\
    0 & \text{else.}
\end{cases}
\]

\[
b_{P,C} \in [0, 1]
\]

- Large margin means very robust, smaller margin less robust
- Connection to performance
The Main Tools

- Generalized Stability Margin
  - Genuinely MIMO
  - Extends SISO single-loop measures from legacy systems

\[ b_{P,C} = 0.3 \sim \text{Gain Margin}=2.7\text{dB, Phase Margin}=15\ \text{deg} \]

- Not really true and still needs validation
- For us,
  - \( b_{P,C} \) and \( b_{P,C,\omega} \) may be computed from experimental data using frequency-domain system identification
  - Requires the estimation of a transfer function infinity-norm or maximal singular value
  - Even for SISO systems, \( b_{P,C} \) is a MIMO calculation
The Main Tools

- **Vinnicombe $\nu$-gap metric**

  \[
  \delta_\nu(C_1, C_2) = \begin{cases} 
  \left\| \frac{1}{(I + C_2^*C_2)^{\frac{1}{2}}(C_1 - C_2)(I + C_1^*C_1)^{\frac{1}{2}}}_\infty \right. & \text{if WNC holds} \\
  1, & \text{else}
  \end{cases}
  \]

  - A true metric measuring the distance between MIMO transfer functions
  - A scaled variant on the distance between Nyquist plots in SISO case
  - A similar condition involving $P$ and $C$ arises in MIMO Nyquist stability
  - Frequency-by-frequency variant

- **WNC**

  \[
  \text{WNC:} \begin{cases} 
  \det(I + C_2^*C_1)(j\omega) \neq 0, \forall \omega, \text{ and} \\
  \text{wno det}(I + C_2^*C_1) + \eta(C_1) - \eta(C_2) = 0
  \end{cases}
  \]

- **Frequency-by-frequency variant**

  \[
  \delta_\nu(C_1, C_2, \omega) = \begin{cases} 
  \sigma_{\text{max}} \left[ \left( I + C_2^*C_2 \right)^{\frac{1}{2}}(C_1 - C_2)(I + C_1^*C_1)^{\frac{1}{2}} \right](j\omega), & \text{if WNC holds} \\
  1, & \text{else}
  \end{cases}
  \]
The Main Tools

- **Vinnicombe $\nu$-gap metric**
  - Easily computable via matlab "gapmetric.m"
  - $\delta_\nu(C_1, C_2) \in [0, 1]$
  - Large $\nu$-gap means large distance
  - Inherently MIMO quantity
  - Extends the Gap Metric into an easily computable value
  - Familiar
    - Almost intuitive
Main Inferences

Vinnicombe’s stability and margin guarantees

- $(P, C_2)$ is stable, if $(P, C_1)$ is stable and $\delta_\nu(C_1, C_2) < b_{P,C_1}$

- $\arcsin b_{P,C_2} \geq \arcsin b_{P,C_1} - \arcsin \delta_\nu(C_1, C_2)$

- $b_{P,C} \in [0, 1]$  \quad $\delta_\nu(C_1, C_2) \in [0, 1]$

Frequency-by-frequency version

- $\arcsin b_{P,C_2,\omega} \geq \arcsin b_{P,C_1,\omega} - \arcsin \delta_\nu(C_1, C_2, \omega)$

- $b_{P,C_1,\omega}$ is experimentally measurable
- $\delta_\nu(C_1, C_2, \omega)$ is easily computable
- $b_{P,C_2,\omega}$ can be inferred
A controller is said to be **certified** if its generalized stability margin is guaranteed at a sufficient level in operation with the plant.

- The \( \nu \)-gap metric and \( b_{PC} \) are the central tools.
  - inherently able to handle MIMO system
  - frequency-by-frequency test
  - easy computability – “gapmetric.m”

**Starting Point**
- Single uncertain engine, \( P \), with a fixed model, \( \hat{P} \)
- A **large set**, \( \mathcal{C} \), of candidate controllers, \( C_i \)
- Every \( C_i \) is designed to achieve internal stability with \( \hat{P} \)
- **Selection** of a **small subset**, \( \{C_j\} \subset \mathcal{C} \), so that through experimental testing of pairs \((P, C_j)\), the complete set \( \mathcal{C} \) is certified.
Mathematical Definitions

- A controller $C$ is said to be **certified** at level $\alpha$ if, using experimental data with the unknown actual plant $P$, we can guarantee
  \[ b_{p,c} > \alpha. \]

- A controller $C$ is said to be **rejected** at level $\beta$, if we can guarantee
  \[ b_{p,c} \leq \beta. \]
Stability / Margin Guarantees

- **Main Inferences**
  - Vinnicombe’s stability and margin guarantees
    - \((P, C_2)\) is stable, if \((P, C_1)\) is stable and \(\delta_\nu(C_1, C_2) < b_{P,C_1}\)
      \[\arcsin b_{P,C_2} \geq \arcsin b_{P,C_1} - \arcsin \delta_\nu(C_1, C_2)\]

- **Refined Inferences**
  - We can infer \(b_{P,C_2} > \alpha\) from \(b_{P,C_1} > \alpha\) and
    \[\arcsin \delta_\nu(C_1, C_2) < \arcsin b_{P,C_2} - \arcsin \alpha\]
    - This requires no additional test on \((P, C_2)\)
  - We can guarantee \(b_{P,C_2} \leq \beta\) from \(b_{P,C_1} \leq \beta\), without extra test, if
    \[\arcsin \delta_\nu(C_1, C_2) < \arcsin \beta - \arcsin b_{P,C_1}\]
Certification Algorithm (Finite Controller Set)

- **Step 1 (Search)** For each uncertified controller, $C_i$, with $b_{P,C_i} > \alpha$
  - count the number of uncertified controllers, $C_j$, that satisfy
    \[
    \arcsin \delta_{\nu}(C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha
    \]
  - Then choose the controller, $C_i$, with most controllers, $C_j$, satisfying this

- **Step 2 (Experiment)** Do the experiment on $(P, C_i)$ to retrieve $b_{P,C_i}$ from closed-loop data

- **Step 3A (Certifying)** If $b_{P,C_i} > \alpha$, certify all controllers, $C_j$, satisfying
  \[
  \arcsin \delta_{\nu}(C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha
  \]

- **Step 3B (Reject)** If $b_{P,C_i} \leq \beta$, reject the controllers, $C_j$, satisfying
  \[
  \arcsin \delta_{\nu}(C_i, C_j) \leq \arcsin \beta - \arcsin b_{P,C_i}
  \]

- Iterate from Step 1 to Step 3 until all controllers are certified or rejected.
Numerical Examples

- Parametrized controllers, $C = \frac{\rho_1 z}{z + \rho_2}$

$$P(z) = \frac{0.1z^2(z - 0.3)}{(z - 0.8)(z - 0.2 + 0.9i)(z - 0.2 - 0.9i)(z - 0.6 + 0.6i)(z - 0.6 - 0.6i)}$$

$$\hat{P}(z) = \frac{0.003z^2 + 0.0801z + 0.1259}{z^3 - 1.1229z^2 + 1.0135z - 0.3940}$$

$\delta_v (P(z), \hat{P}(z)) = 0.1917$ (Unknown)
Numerical Examples

- Parametrized controllers, \( C = \frac{\rho_1 z}{z + \rho_2} \)

\[
P(z) = \frac{0.1z^2(z - 0.3)}{(z - 0.8)(z - 0.2 + 0.9i)(z - 0.2 - 0.9i)(z - 0.6 + 0.6i)(z - 0.6 - 0.6i)}
\]

\[
\hat{P}(z) = \frac{0.003z^2 + 0.0801z + 0.1259}{z^3 - 1.1229z^2 + 1.0135z - 0.3940}
\]

Not a stabilizing region for the actual plant

Stabilizing Region of the Nominal Model, \( \hat{P} \).

Stabilizing Region of the Actual Plant, \( P \).
(Unknown)

\( \delta_v (P(z), \hat{P}(z)) = 0.1917 \) (Unknown)

Stabilizing regions for \( P \), \( \hat{P} \) and 74 selected controller parameters

ETH, April 3, 2008
Numerical Examples (stability certification)

1. **(Search)** For each uncertified controller, \( C_i \), count the number of uncertified controllers, \( C_j \), satisfying

\[
\arcsin \delta_{\nu}(C_i, C_j) < \arcsin b_{P,C_i}
\]  

(1)

If we choose \( C_{37} \) as a test controller, then we could expect there are 34 controllers satisfying

\[
\arcsin \delta_{\nu}(C_{37}, C_j) < \arcsin b_{\hat{P},C_{37}}
\]

which means \( b_{\hat{P},C_j} > 0 \) for those controllers \( C_j \)
Numerical Examples (stability certification)

2. **(Experiment)** Do the experiment on the plant-controller pair \((P, C_{37})\) to retrieve \(b_{P,C_{37}}\) from closed-loop data.

3. **(Certifying)** Actually 21 controllers satisfy

\[
\arcsin \delta_\nu(C_{37}, C_j) < \arcsin b_{P,C_{37}}
\]

21 certified controllers by the first experiment
Numerical Examples (stability certification)

- Iteration for stability certification
- After 31 tests, stability certification for 60 controllers (blue circles) was completed.

\[ \arcsin \delta_L(C_i, C_j) < \arcsin b_{P,C_i} \]
Numerical Examples (margin certification)

- 27 controllers certified as $b_{P,C} > 0.3$, by 7 experiments.
Conservatism in Computations

- The generalized stability margin and the $\nu$-gap metric computations involve the maximum singular value over all frequency points
  - A supremum over frequency
  - A scalar measure of a matrix property
Conservatism in Computations

- The generalized stability margin and the $\nu$-gap metric computations involve the maximum singular value over all frequency points
  - A supremum over frequency $\Rightarrow$ frequency-by-frequency analysis
  - A scalar measure of a matrix property
Conservatism in Computations

- The generalized stability margin and the $\nu$-gap metric computations involve the maximum singular value over all frequency points
  - A supremum over frequency $\Rightarrow$ frequency-by-frequency analysis
  - A scalar measure of a matrix property $\Rightarrow$ input and output scalings
Simultaneous Scalings

- For an efficient controller certification, larger $b_{P,C_i}$ and smaller $\delta_v (C_i, C_j)$ should be incorporated if possible.

\[
\arcsin \delta_v (C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha
\]

\[
\iff b_{P,C_j} > \alpha
\]
Simultaneous Scalings

- For an efficient controller certification, **larger** $b_{P,C_i}$ and **smaller** $\delta_{v}(C_i, C_j)$ should be incorporated if possible.

\[
\arcsin \delta_{v}(C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha \\
\iff b_{P,C_j} > \alpha
\]

\[
\arcsin \delta_{v}(W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1}) < \arcsin b_{W_oPW_i, W_i^{-1}C_iW_o^{-1}} - \arcsin \alpha \\
\iff b_{W_iPW_o, W_i^{-1}C_jW_o^{-1}} > \alpha
\]
Simultaneous Scalings

- For an efficient controller certification, larger $b_{P,C_i}$ and smaller $\delta_v(C_i, C_j)$ should be incorporated if possible.

$$\arcsin \delta_v(C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha$$

$$\iff b_{P,C_j} > \alpha$$

$$\arcsin \delta_v(W^{-1}_i C_i W^{-1}_o, W^{-1}_i C_j W^{-1}_o) < \arcsin b_{W_o PW_i, W^{-1}_i C_i W^{-1}_o} - \arcsin \alpha$$

$$\iff b_{W_i PW_o, W^{-1}_i C_j W^{-1}_o} > \alpha$$
Simultaneous Scalings

- For an efficient controller certification, larger $b_{P,C_i}$ and smaller $\delta_v (C_i, C_j)$ should be incorporated if possible.

\[
\arcsin \delta_v (C_i, C_j) < \arcsin b_{P,C_i} - \arcsin \alpha \quad \iff \quad b_{P,C_j} > \alpha
\]

\[
\arcsin \delta_v (W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1}) < \arcsin b_{W_oPW_i, W_i^{-1}C_iW_o^{-1}} - \arcsin \alpha \quad \iff \quad b_{W_iPW_o, W_i^{-1}C_jW_o^{-1}} > \alpha
\]
Problem Setting

- $W_i, W_o$ positive definite symmetric matrices, constant over frequency $\omega \in \Omega$

- $W_i, W_o$ positive definite hermitian matrices at a fixed frequency $\omega_n$

- $W_i(s), W_o(s)$ bi-stable and bi-proper transfer function matrices interpolating $W_i, W_o$ at a sequence of frequency values $\{\omega_n\}$
Problem Setting

- Scaling for $b_{\hat{\mathcal{P}},C}$ computation: **convex optimization**
- Scaling for $\nu$ - gap computation: **non-convex optimization**
- **Simultaneous scaling selections for** $b_{\hat{\mathcal{P}},C}$ **and** $\nu$ - gap
  (XY-Centering algorithm, Skelton and Iwasaki)

  \[
  \text{Increase } b_{W_o\hat{\mathcal{P}}_i,W_i^{-1}C_iW_o^{-1}}
  \]

  \[
  \text{Decrease } \delta_{\nu} \left( W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1} \right)
  \]

- Stability is invariant to scalings

  \[
  b_{W_o\hat{\mathcal{P}}_i,W_i^{-1}C_iW_o^{-1}} > 0 \quad \text{if and only if} \quad b_{\hat{\mathcal{P}},C} > 0
  \]

- Scalings have a stability analysis purpose (In SISO systems, GM/PM is invariant to scalings)
F100 Jet Engine Certification – Use of Scalings

- Constant diagonal input and output scalings \((W_i, W_o)\) for less conservative margin computations

\[
\arcsin \delta (W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1}) < \arcsin b W_o \hat{P} W_i, W_i^{-1} C_i W_o^{-1} - \arcsin \alpha
\]

- The F100 engine model was chosen at sea level static flight conditions with power lever angle (PLA) of 36 degree
- Reduced order model is used in controller design
- The model-stabilizing multivariable Proportional-plus-Integral controller is designed in the LQR framework

\[
C(s) = K_p + \frac{1}{s} K_I
\]

where, \(K_p\) and \(K_I\) are 5x5 gain matrices

- The first certification parameter is \(K_p(1,1)\) and the second one is \(K_I(2,2)\)
- Each parameter changes up to \(\pm 30\%\) from its original value
a), b): 8 tests complete certification of 81 controllers for $b_{p,c} > 0.2$

b) c), d): all controllers are rejected at level of $\alpha = 0.23$
Implementation of Controller Certification

- Development of GUI in corporation with SC Solutions
I. Controller Certification

II. Estimation of the Generalized Stability Margin
I. Controller Certification

II. Estimation of the Generalized Stability Margin

ETH, April 3, 2008

Tuesday, April 3, 2012
Introduction to the $b_{P,C}$ estimation

- The stability margin inference of $b_{P,C_2}$ relies on the estimated value of $b_{P,C_1}$
  \[
  \arcsin b_{P,C_2} \geq \arcsin b_{P,C_1} - \arcsin \delta_v (C_1, C_2)
  \]

- The generalized stability margin will be estimated from the empirical transfer function estimate (ETFE) of $T(P,C)$
  \[
  y(t) = T(P,C)u(t) + \nu(t).
  \]
  \[
  T(P,C) = \begin{bmatrix}
  P(I + CP)^{-1}C & P(I + CP)^{-1} \\
  (I + CP)^{-1}C & (I + CP)^{-1}
  \end{bmatrix}, \quad b_{P,C} = \|T(P,C)\|^{-1}_\infty
  \]

- A scalar standard test signal for the error bound analysis – a probing input signal over a specific range of frequency, $[\omega_1, \omega_2]$.

- The same scalar test signal used in the experiment design for the MIMO ETFE.
Estimating Frequency Response

- Estimating one frequency response value - SISO case
  - Input / Output \[ \{ u(t) \mid t = -N_r, \ldots, 0, \ldots, N - 1 \} \]
  \[ \{ y(t) \mid t = X, \ldots, X, 0, \ldots, N - 1 \} \]

- Model
  \[ y(t) = \sum_{n=0}^{\infty} g(n)u(t-n) + v(t) = T(z)u(t) + v(t) \]

- DFTs
  \[ U_N^l = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} u(t)e^{-j\frac{2\pi t}{N}}, \quad l = 0, 1, \ldots, N - 1 \]
  \[ Y_N^l = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(t)e^{-j\frac{2\pi t}{N}}, \quad l = 0, 1, \ldots, N - 1 \]

- Empirical Transfer Function Estimate
  \[ \hat{T}\left(e^{j\frac{2\pi l}{N}}\right) = \frac{Y_N^l}{U_N^l}, \quad l = 0, 1, \ldots, N - 1 \]

- SISO Error Analysis

\[ T\left(e^{j\frac{2\pi l}{N}}\right) - \hat{T}\left(e^{j\frac{2\pi l}{N}}\right) \leq \frac{1}{\sqrt{N}\left|U_N^l\right|} \left( u_{\text{max}} + \bar{u} \right) \frac{M\rho^2}{(1-\rho)^2} \rho^{N_r} (1-\rho^N) + \frac{\bar{V}_N^l}{U_N^l} \]
SISO Error Analysis

\[ \left| T \left( e^{\frac{2\pi i}{N}} \right) - \hat{T} \left( e^{\frac{2\pi i}{N}} \right) \right| \leq \frac{1}{\sqrt{N}|U_N^l|} \left( u_{\text{max}}^* + \bar{u} \right) \frac{M \rho^2}{(1 - \rho)^2} \rho^N (1 - \rho^N) + \frac{V_N^l}{|U_N^l|} \]

- Standard Test Signal \( u(t), \ t = -N_r, \ldots, -1, 0, \ldots, N - 1 \)
- Pre-experiment (\( N_r \)) to reduce the initial conditions (De Vries, 1994)
- Chirp up & down (\( N \))
  - manageable probing frequency range (wide then narrow)
  - smooth (not multi-sine)
  - \( |U_N^l| \neq 0 \) for \([\omega_1, \omega_2]\)
- \textbf{a priori} information
  - Inputs:
    \[ |u(t)| \leq \begin{cases} u_{\text{max}}, & \text{for } t \in [-N_r, N - 1] \\ \bar{u}, & \text{for } t < -N_r \end{cases} \]
  - Impulse response: \(|g(k)| \leq M \rho^k\) with \( M > 0, 0 < \rho < 1 \)
  - Noise power: \( \frac{V_N^l}{U_N^l} \)

Probing from DC to 5 Hz
SISO Error Analysis

\[
\left| T\left( e^{\frac{j2\pi l}{N}} \right) - \hat{T}\left( e^{\frac{j2\pi l}{N}} \right) \right| \leq \frac{1}{\sqrt{N}} \frac{u^{\text{max}} + \bar{u}}{U_N^l} \frac{M \rho^2}{(1 - \rho)^2} \rho^{N_r} (1 - \rho^N) + \frac{V_N^l}{U_N^l}
\]

- Dependencies
  - Number of data \( N \)
  - Ringdown time of engine \( \rho \), impulse response bound \( M \)
  - Number of “pre-experiment” of the standard test excitation \( N_r \)
  - Input energy in band \( \left| U_N^l \right| \)
  - Noise energy in band \( \left| V_N^l \right| \)
  - Bounds on pre-input and test inputs \( u^{\text{max}}, \bar{u} \)
SISO to MIMO

- If $m$ input signals, then need to perform $m$ separate experiments
  - Each has its own pre-experiment
  - Define an $m \times m$ matrix $Q = [\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_m]$
  - Each has the vector input signal $\tilde{u}_i(t) = \tilde{q}_i u(t)$, $i = 1, 2, \cdots, m$
    - Same scalar standard test signal $u(t)$
  - Matrix $Q$ of all experiments has full rank
  - Empirical Transfer Function Estimate (MIMO)

$$
T \left( e^{j \frac{2\pi l}{N}} \right) = Y_N \left( U^l \right)^{-1} = [Y_{N1}^l, Y_{N2}^l, \cdots, Y_{Nm}^l][U_{N1}^l, U_{N2}^l, \cdots, U_{Nm}^l]^{-1} = [\tilde{Y}_{N1}^l, \tilde{Y}_{N2}^l, \cdots, \tilde{Y}_{Nm}^l](QU_N^l)^{-1}
$$

where $\tilde{Y}_{Ni}^l, \tilde{U}_{Ni}^l$ ($i = 1, 2, \cdots, m$) are the vector DFTs of output vectors, $\tilde{u}_i(t)$, and input vectors, $\tilde{u}_i(t)$, respectively.

- MIMO error bound

$$
\tilde{\sigma} \left( T \left( e^{j \frac{2\pi l}{N}} \right) - \hat{T} \left( e^{j \frac{2\pi l}{N}} \right) \right) \leq \frac{1}{\sqrt{N} |U_N^l|} \frac{\tilde{\sigma}(Q)}{\sigma(Q)} \left( u^\text{max} + \overline{u} \right) \frac{\tilde{M} \tilde{\rho}^2}{(1 - \tilde{\rho})^2} \tilde{\rho}^N \tilde{\rho}^N + \frac{\tilde{V}_N^l}{|U_N^l| \sigma(Q)}
$$

ETH, April 3, 2008

Tuesday, April 3, 2012
Single Frequency Result to Neighboring Frequencies Bound

- In order to bound the error between two fixed frequency points
- Interpolation error and dependencies

\[
\sigma \left( T \left( e^{\frac{2\pi (l+r)}{N}} \right) - T \left( e^{\frac{2\pi l}{N}} \right) \right) \leq \frac{\tilde{M} \tilde{\rho}}{(1 - \tilde{\rho})^2} \frac{\pi}{N}
\]

with \( l = 0, 1, \ldots, N - 1 \) and \( -\frac{1}{2} \leq r \leq \frac{1}{2} \).

\[
\| T(z) \|_\infty \leq \max_{k \in [0, N-1]} \left\{ \sigma \left( \hat{T} \left( e^{\frac{2\pi l}{N}} \right) \right) + \frac{\tilde{V}}{U_N} \sqrt{\frac{\sigma(Q)}{U_N}} + \frac{\tilde{M} \tilde{\rho}}{(1 - \tilde{\rho})^2} \left( \frac{\pi}{N} + \frac{1}{\sqrt{N}} \frac{\sigma(Q)}{U_N} \right) \left( u_{\max} + \tilde{u} \right) \tilde{\rho}^{N+1} (1 - \tilde{\rho}^N) \right\}
\]
Estimating $b_{P,C}$

- Conduct closed-loop identification experiment for MIMO transfer function $\mathbf{T} (P, C, \omega)$

$$\hat{\mathbf{T}} (P, C) (e^{\frac{j2\pi l}{N}}) = \mathbf{Y}_N^l (\mathbf{U}_N^l)^{-1}, \ l = 0, 1, \ldots, N - 1$$

- Scale at each frequency using earlier results (optional for certification)

$$\hat{\mathbf{T}} (P, C) (e^{\frac{j2\pi l}{N}}) \Rightarrow \mathbf{T} (W_o PW_i, W_i^{-1} CW_o^{-1}) (e^{\frac{j2\pi l}{N}})$$

- Take max value plus bound

$$b_{P,C} \geq \frac{1}{\max_{k \in [0, N-1]} \left[ \bar{\sigma} \left( \hat{\mathbf{T}} (P, C) (e^{\frac{j2\pi l}{N}}) \right) + \Delta T_l \right]}$$

where

$$\Delta T_l = \frac{\tilde{M} \tilde{\rho}}{(1 - \tilde{\rho})^2} \left( \frac{\pi}{N} + \frac{1}{\sqrt{N} |U_N^l|} \bar{\sigma} (Q) (u_{\text{max}} + \bar{u}) \tilde{\rho}_{N_r+1} (1 - \tilde{\rho}_N) \right) + \frac{\bar{V}_N^l}{|U_N^l| \bar{\sigma} (Q)}$$
Numerical Example (Experiment Design)

- Linearized MAPSS Model: \( T(\hat{P}, C, \omega) \), a \( 6 \times 6 \) transfer function matrix
- Using \( 6 \times 6 \) Hadamard matrix, \( Q \), and a standard test input, \( u(t) \).

\[
Qu(t) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 \\
\end{bmatrix} u(t)
\]

6 consecutive experiments with the standard test inputs

\[
\begin{bmatrix}
\hat{y}_1(t) & \hat{y}_2(t) & \hat{y}_3(t) & \hat{y}_4(t) & \hat{y}_5(t) & \hat{y}_6(t)
\end{bmatrix} = T(\hat{P}, C)\begin{bmatrix}
\bar{u}_1(t) & \bar{u}_2(t) & \bar{u}_3(t) & \bar{u}_4(t) & \bar{u}_5(t) & \bar{u}_6(t)
\end{bmatrix} = T(\hat{P}, C)Qu(t)
\]

6 consecutive standard test inputs for the first channel
Numerical Example

- Matrix DFT: \( \hat{T} (P, C) (e^{\frac{2\pi j}{N}}) = Y_N^l (U_N^l)^{-1}, \ l = 0, 1, \ldots, N - 1 \)

The maximum singular value plot of  a)\( T (P, C) \) and b)ETFE of \( \hat{T} (P, C) \)

\[
\| T (P, C) \|_\infty = 70.0884 \text{ db at } \omega = 1.8406 \text{ rad / sec} \\
\| \hat{T} (P, C) \|_\infty = 70.0195 \text{ db at } \omega = 1.8408 \text{ rad / sec}
\]
Numerical Example

- Effect of different length of pre-experiments $N_r$

| Length of the Pre-experiments | $||\hat{T}(P,C)||_\infty$ | $||T(P,C)||_\infty - ||\hat{T}(P,C)||_\infty$ |
|------------------------------|---------------------|------------------------------------------|
| $0 \times u(t)$              | 81.0507 dB          | 10.9623 dB                               |
| $1/32 \times u(t)$           | 73.5809 dB          | 3.4925 dB                                |
| $1/10 \times u(t)$           | 70.0196 dB          | 0.0688 dB                                |
| $1/2 \times u(t)$            | 70.0881 dB          | 0.0003 dB                                |
Noise Reduction in ETFE

- Downsampling the ETFE
  - Take $N=MP$ samples of data, $i=0,1,...,MP-1$
  - Compute the $N$-point DFTs and ETFE
  - Extract every $M$th point, $i=0, M-1, 2M-1,..., MP-1$ from the ETFE
  - The effective noise power at these $M$ frequencies is reduced by a factor $P$
    - ETFE estimation error is improved at each frequency
    - Interpolation error is increased because we have ETFE samples only at $M$ points
  - Tradeoff
Conclusions

- Development of the stability inference tool
  - Identifying the \( \nu \)-gap metric as a MIMO controller certification tool
  - Providing certification/rejection algorithms

- LMI formulations for the generalized stability margin and the \( \nu \)-gap metric computations for efficient MIMO controller certification
  - Reducing conservatism in computations of margin and metric individually and jointly through scalings

- Experiment design for estimation of the generalized stability margin with an guaranteed error bound
  - Derivation of the error bound
  - Designing inputs for MIMO ETFE (pre-experiments)
Further Work

- Certification algorithm of an infinite set of MIMO controllers
- Certification for fleet variability
- Time variation in plants and/or controllers
- A direct method to certify a non-linear plant and controller pair
- Multi-objective optimization for maximizing the difference between the scaled generalized margin and $\nu$-gap metric
- Methodology for retrieving the upper bound on impulse response empirically
Implementation of Controller Certification

- Development of GUI in corporation with SC Solutions
Acknowledgment

- Certification for JSF engine

DoD

SBIR

F135 test stand

Pratt & Whitney

Impacts

SC Solutions

Lockheed Martin

Test data
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Impact

SC Solutions

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Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

**Theorem 3** At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_o$ with $X_i = Y_i^{-1} = W_i W_i^*$ and $X_o = Y_o^{-1} = W_o W_o^*$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P}, C_i^*)(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P, C_i)(j\omega_n) > 0,$$

$$\left[ \gamma_2^2 (X_i + C_j Y_o C_j^*) (C_j - C_i) (C_j - C_i)^* (X_o + C_i^* Y_i C_i) \right] (j\omega_n) > 0$$

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} > 0$$

with following properties,

$$\begin{cases} (X_i^{-1}, X_o) \in \Phi(\gamma_1) \\ (X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_1) \end{cases} \quad \text{or} \quad \begin{cases} (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\ (Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2) \end{cases}$$

Then the scaled $b_{P,C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$,

$$b_{W_i \hat{P} W_o, W_i^{-1} C_j W_o^{-1}}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_{\nu}(W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2$$
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

**Theorem 3** At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_o$ with $X_i = Y_i^{-1} = W_i^{-1}$ and $X_o = Y_o^{-1} = W_o^{-1}$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$
\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P}, C_i)(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P, C_i)(j\omega_n) > 0,
$$

$$
\gamma_2^2 \begin{bmatrix} X_i + C_j Y_o C_j^* & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0
$$

with following properties,

$$
\begin{cases}
(X_i^{-1}, X_o) \in \Phi(\gamma_1) \\
(X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2)
\end{cases}
$$

or

$$
\begin{cases}
(Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\
(Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2)
\end{cases}
$$

Then the scaled $b_{P,C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$

$$
b_{\hat{P}W_i, W_o^{-1}C_jW_o^{-1}}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_{\nu}(W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1})(j\omega_n) < \gamma_2
$$
Simultaneous Scaling of $b_{P, C}$ and $\nu$-gap

**Theorem 3**  At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_0$ with $X_i = Y_i^{-1} = W_i W_i^*$ and $X_o = Y_o^{-1} = W_o W_o^*$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

\[
\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T\left(\hat{P}, C_i\right)^*(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T\left(P, C_i\right)(j\omega_n) > 0,
\]

\[
\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0
\]

\[
\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0
\]

with following properties,

\[
\left\{ \begin{array}{l}
(X_i^{-1}, X_o) \in \Phi(\gamma_1) \\
(X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2)
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l}
(Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\
(Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2)
\end{array} \right.
\]

Then the scaled $b_{P, C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$

\[
b_{W_i \hat{P} W_o, W_i^{-1} C_j W_o^{-1}}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_\nu (W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2
\]
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

**Theorem 3** At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_o$ with $X_i = Y_i^{-1} = W_i W_i^*$ and $X_o = Y_o^{-1} = W_o W_o^*$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T\left(\hat{P}, C_i\right)^*(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T\left(P, C_i\right)(j\omega_n) > 0,$$

$$\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0$$

with following properties,

$$\left\{ \begin{array}{l} (X_i^{-1}, X_o) \in \Phi(\gamma_1) \\ (X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2) \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\ (Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2) \end{array} \right\}$$

Then the scaled $b_{P,C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$.

$${b_{W_i, \hat{P} W_o, W_i^{-1} C_j W_o^{-1}}(j\omega_n)} > \gamma_1^{-1} \quad \text{and} \quad \delta_{\nu}(W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2$$
Numerical Example

<table>
<thead>
<tr>
<th>Scaling</th>
<th>$b_{\hat{P},C_0}$</th>
<th>$\delta_v(C_0, C_1)$</th>
<th>$b_{\hat{P},C_1} \left( \text{sin}(\arcsin b_{\hat{P},C_0} - \arcsin \delta_v(C_0, C_1)) \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without</td>
<td>0.5495</td>
<td>0.5407</td>
<td>&gt; 0.0105</td>
</tr>
<tr>
<td>$b_{\hat{P},C_0}^{-1}$ only</td>
<td>0.9964</td>
<td>0.9921</td>
<td>&gt; 0.0409</td>
</tr>
<tr>
<td>$b_{\hat{P},C_0}^{-1}$ and $\delta_v(C_0, C_1)$</td>
<td>0.6362</td>
<td>0.4521</td>
<td>&gt; 0.2188</td>
</tr>
</tbody>
</table>

- Scalings for maximizing

\[
W_o = \begin{bmatrix}
1.3159 & 1.0086 - 0.0016i \\
1.0086 + 0.0016i & 1.3073
\end{bmatrix}, \quad W_i = \begin{bmatrix}
0.2088 & 0.07 - 0.0025i \\
0.07 + 0.0025i & 0.3388
\end{bmatrix}
\]

- Simultaneous scalings increased the generalized stability margin and decreased the $\delta_v$-gap
Frequency Dependent Scalings of $b_{P,C}$

**Theorem 2.1** At a fixed frequency $\omega_n$, consider positive definite hermitian matrices $W_o$ and $W_i$ with $Y_i = (W_i W_i^*)^{-1}$ and $X_o = W_o^* W_o$. If a solution $(X_o, Y_i)$ of the following optimization,

$$\begin{align*}
\text{minimize} & \quad \gamma_1^2 \\
\text{subject to} & \quad \gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(P, C_i)^* (j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P, C_i)(j\omega_n) > 0,
\end{align*}$$

exists with achieved objective value $\gamma_1$, then the scaled $b_{P,C_i}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$,

$$b_{W_i P W_o, W_i^{-1} C_j W_o^{-1}} (j\omega_n) > \gamma_1^{-1}$$
Frequency Dependent Scalings of $b_{p,c}$

**Theorem 2.1** At a fixed frequency $\omega_n$, consider positive definite hermitian matrices $W_o$ and $W_i$ with $Y_i = (W_iW_i^*)^{-1}$ and $X_o = W_o^*W_o$. If a solution $(X_o, Y_i)$ of the following optimization,

$$
\min_{Y_i,X_o} \gamma_1^2 \left[ \begin{array}{cc} X_o & 0 \\ 0 & Y_i \end{array} \right] - T(P,C_i)^*(j\omega_n) \left[ \begin{array}{cc} X_o & 0 \\ 0 & Y_i \end{array} \right] T(P,C_i)(j\omega_n) > 0,
$$

exists with achieved objective value $\gamma_1$, then the scaled $b_{\tilde{p},c_i}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$.

$$
b_{W_iPW_o^{-1}W_i^{-1}C_jW_o^{-1}}(j\omega_n) > \gamma_1^{-1}
$$

ETH, April 3, 2008
Theorem 2.1  At a fixed frequency $\omega_n$, consider positive definite hermitian matrices $W_o$ and $W_i$ with $Y_i = (W_iW_i^*)^{-1}$ and $X_o = W_o^*W_o$. If a solution $(X_o, Y_i)$ of the following optimization,

\[
\begin{align*}
\text{minimize} & \quad \gamma_1^2 Y_iX_o \\
\text{subject to} & \quad \gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(P, C_i) (j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P, C_i) (j\omega_n) > 0, \\
& \quad X_o > 0, \ Y_i > 0
\end{align*}
\]

exists with achieved objective value $\gamma_1$, then the scaled $b_{P,C_i}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$.

\[
b_{W_iPW_oW_i^{-1}C_jW_o^{-1}} (j\omega_n) > \gamma_1^{-1}
\]
**Constant Diagonal Scalings of** $b_{P,C}$

**Corollary 1.** Consider constant positive definite hermitian matrices $W_i$ and $W_o$ with $M = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^{-1} W_i^{-1} \end{bmatrix}$ and let $(A,B,C,D)$ be a state space realization of $T(\hat{P},C)$. If a solution $M$ of LMI optimization, 

\[
\text{minimize } \gamma_1^2 \\
\text{subject to } \begin{bmatrix} C^* \\ D^* \end{bmatrix} M [C \ D] + \begin{bmatrix} QA + A^* Q & QB \\ B^* Q & 0 \end{bmatrix} < \gamma_1^2 \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}
\]

exists with achieved objective value $\gamma_1$, then the scaled $b_{\hat{P},C}$ is bounded below by $\gamma_1^{-1}$

\[
b_{W_i \hat{P} W_o, W_i^{-1} C_j W_o^{-1}}(j\omega_n) > \gamma_1^{-1}
\]

over all frequencies $\omega \in \mathfrak{i}$.
Corollary 1. Consider constant positive definite hermitian matrices $W_i$ and $W_o$ with $M = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^{-1} W_i^{-1} \end{bmatrix}$ and let $(A, B, C, D)$ be a state space realization of $T(\hat{P}, C)$. If a solution $M$ of LMI optimization,

\[
\begin{align*}
\text{minimize} & \quad \gamma_1^2 \\
\text{subject to} & \quad \begin{bmatrix}
C^* \\
D^*
\end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} QA + A^* Q & QB \\ B^* Q & 0 \end{bmatrix} < \gamma_1^2 \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}
\end{align*}
\]

exists with achieved objective value $\gamma_1$, then the scaled $b_{\hat{P}, C}$ is bounded below by $\gamma_1^{-1}$

\[
b_{W_i, \hat{P} W_o, W_i^{-1} C, W_o^{-1}}(j \omega_n) > \gamma_1^{-1}
\]

over all frequencies $\omega \in \mathbb{i}$.

ETH, April 3, 2008
Constant Diagonal Scalings of $b_{p,c}$

**Corollary 1.** Consider constant positive definite hermitian matrices $W_i$ and $W_o$ with $M = \begin{bmatrix} W_o^*W_o & 0 \\ 0 & W_i^{-1}W_i^{-1} \end{bmatrix}$ and let $(A, B, C, D)$ be a state space realization of $T(\hat{P}, C)$. If a solution $M$ of LMI optimization,

\[
\begin{align*}
\text{minimize} & \quad \gamma_1^2 \\
\text{subject to} & \quad \begin{bmatrix} C^* \\ D^* \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} QA + A^*Q & QB \\ B^*Q & 0 \end{bmatrix} < \gamma_1^2 \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \\
M & > 0, \quad Q = Q^* 
\end{align*}
\]

exists with achieved objective value $\gamma_1$, then the scaled $b_{\hat{P}, C}$ is bounded below by $\gamma_1^{-1}$

\[
b_{W_i, \hat{P}W_o, W_i^{-1}C, W_o^{-1}}(j\omega_n) > \gamma_1^{-1}
\]

over all frequencies $\omega \in \mathbb{I}$. 

ETH, April 3, 2008
Scaled $\nu$ -gap metric

- Scaled $\nu$-gap

$$
\delta_\nu (W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1})
= \left\| \left( I + W_i^{-1}C_jW_o^{-1}W_o^*-1C_i^*W_i^*-1 \right)^{-1/2} W_i^{-1} (C_j - C_i) W_o^{-1} \left( I + W_o^*-1C_i^*W_i^*-1W_i^{-1}C_iW_o^{-1} \right)^{-1/2} \right\|_\infty
$$

Using $\|A\|_\infty = \sup_{\omega \in \mathbb{I}} \lambda^{1/2}_{\max} (A^*A)$ and $\lambda_{\max} (AB) = \lambda_{\max} (BA)$, we have

$$
\sup_{\omega \in \mathbb{I}} \lambda^{1/2}_{\max} \left( (W_i^{-1}W_i^* + C_j(W_o^*W_o)^{-1}C_j^*)^{-1}(C_j - C_i)(W_o^*W_o + C_i^*(W_iW_i^*)^{-1}C_i)^{-1}(C_j - C_i)^* \right)(j\omega)
$$

Using $X_i = Y_i^{-1} = W_iW_i^*$ and $X_o = Y_o^{-1} = W_o^*W_o$, minimization of the scaled $\nu$-gap can be formulated as

- minimize $\gamma^2$

- subject to

$$
\left( (X_i + C_jY_oC_j^*)^{-1}(C_j - C_i)(X_o + C_i^*Y_iC_i)^{-1}(C_j - C_i)^* \right)(j\omega_n) < \gamma^2 I,
$$

$X_i = Y_i^{-1} > 0, X_o = Y_o^{-1} > 0$
Scaled $\nu$-gap metric

Scaled $\nu$-gap

$$\delta_\nu (W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1})$$

$$= \left\| \left( I + W_i^{-1}C_jW_o^{-1}W_o^{-1}C_jW_i^{-1} \right)^{-1/2} W_i^{-1} \left( C_j - C_i \right) W_o^{-1} \left( I + W_o^{-1}C_iW_i^{-1}W_i^{-1}C_iW_o^{-1} \right)^{-1/2} \right\|_\infty$$

Using $\|A\|_\infty = \sup_{\omega \in \Omega} \lambda_{\max}^{1/2} (A \ast A)$ and $\lambda_{\max}(A \ast B) = \lambda_{\max}(B \ast A)$, we have

$$\sup_{\Omega \in \Omega} \lambda_{\max}^{1/2} \left( \left( W_i^{-1}C_jW_o^{-1} + C_j(W_o^{-1}C_i)^{-1}C_j \right) \left( C_j - C_i \right) \left( W_o^{-1}C_iW_i^{-1} + C_i(W_i^{-1}C_j)^{-1}C_i \right) \left( C_j - C_i \right)^* \right) (j\omega)$$

Using $X_i = Y_i^{-1} = W_i^{-1}W_i$ and $X_o = Y_o^{-1} = W_o^{-1}W_o$, minimization of the scaled $\nu$-gap can be formulated as

minimize $$\gamma_2^2$$

subject to $$\left( \left( X_i + C_jY_oC_j^* \right)^{-1} \left( C_j - C_i \right) \left( X_o + C_iY_iC_i \right)^{-1} \left( C_j - C_i \right)^* \right) (j\omega_n) \leq \gamma_2^2 I,$$

$$X_i = Y_i^{-1} > 0, X_o = Y_o^{-1} > 0$$
LMI Formulation for Scaling $\gamma$-gap

- Using Schur complement,
  Minimize $\gamma_2^2$
  subject to
  $$\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i) & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0,$$
  $$X_i = Y_i^{-1} > 0, X_o = Y_o^{-1} > 0$$

- Minimize $\gamma_2^2$
  subject to
  $$\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i) & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0,$$
  $$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \succeq 0, \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} \succeq 0 \ (X_i \succeq Y_i^{-1}, \ X_o \succeq Y_o^{-1})$$
LMI Formulation for Scaling $\gamma$-gap

- Using Schur complement,
  Minimize $\gamma^2$
  subject to
  $$\begin{bmatrix} \gamma^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i) & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0,$$
  $X_i = Y_i^{-1} > 0, X_o = Y_o^{-1} > 0$

- Minimize $\gamma^2$
  subject to
  $$\begin{bmatrix} \gamma^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i) & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0,$$
  $$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \succeq 0, \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} \succeq 0 \quad (X_i \succeq Y_i^{-1}, \quad X_o \succeq Y_o^{-1})$$
LMI Formulation for Scaling $\gamma$-gap

- Using Schur complement,
  Minimize $\gamma_2^2$
  subject to
  \[
  \begin{bmatrix}
  \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\
  (C_j - C_i) & (X_o + C_i^* Y_i C_i)
  \end{bmatrix}(j \omega_n) > 0,
  \]
  $X_i = Y_i^{-1} > 0$, $X_o = Y_o^{-1} > 0$

- Minimize $\gamma_2^2$
  subject to
  \[
  \begin{bmatrix}
  \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\
  (C_j - C_i) & (X_o + C_i^* Y_i C_i)
  \end{bmatrix}(j \omega_n) > 0,
  \]
  \[
  \begin{bmatrix}
  X_i & I \\
  I & Y_i
  \end{bmatrix} \succeq 0, \quad \begin{bmatrix}
  X_o & I \\
  I & Y_o
  \end{bmatrix} \succeq 0 \quad (X_i \succeq Y_i^{-1}, \quad X_o \succeq Y_o^{-1})
  \]

Iterative minimization
$\lambda_{\max}(X_i Y_i), \quad \lambda_{\max}(X_o Y_o)$
Define the following matrices $\Phi$ and $\Psi$, 

$$
\Phi(Y_i, X_o, \gamma_1) \equiv \gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P}, C_i)(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(\hat{P}, C_i)j\omega_n
$$

$$
\Psi(X_i, Y_i, X_o, Y_o, \gamma_2) = \begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n)
$$

then define two sets $\Phi$ and $\Psi$,

$$
\Phi(\gamma_1) = \{Y_i, X_o : \Phi(Y_i, X_o, \gamma_1) > 0\}
$$

$$
\Psi(\gamma_2) = \{X_i, Y_i, X_o, Y_o : \Psi(X_i, Y_i, X_o, Y_o, \gamma_2) > 0\}
$$
**Theorem 2** At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $P(j\omega)$, stabilizing controllers $C_i(j\omega)$ and $C_j(j\omega)$, and consider positive definite hermitian matrices $W_i$ and $W_o$ with $W_i = W_i^*$ and $W_o = W_o^*$. If a solution exists with achieved objective value for the LMI systems, 

$$\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega) > 0$$

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} > 0$$

with following properties,

$$\begin{cases} (X_i^{-1}, X_o) \in \Phi(\gamma_1) \\ (X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2) \end{cases} \quad \text{or} \quad \begin{cases} (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\ (Y_i^{-1}, Y_i^{-1}, Y_o) \in \Psi(\gamma_2) \end{cases}$$

then the scaled $\nu$-gap metric at $\omega_n$ is bounded above by

$$\delta_\nu (W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2$$
**Theorem 2**  
At a fixed frequency $\omega_n \in \mathbb{R}$, consider the frequency response of a plant model $L(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$, and consider positive definite hermitian matrices $X_i$ and $X_j$ with $X_i = Y^{-1} = W_o W_i$ and $X_j = Y^{-1} = W_i W_i^*$ exists with achieved objective value for the LMI's:

$$
\begin{bmatrix}
\gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\
(C_j - C_i)^* & (X_o + C_i^* Y_i C_i^*)
\end{bmatrix} (j\omega_n) > 0
$$

with following properties,

$$
\begin{cases}
(X_i^{-1}, X_o) \in \Phi(\gamma_1) & \text{or} & (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\
(X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2) & (Y_i^{-1}, Y_o^{-1}, Y_o) \in \Psi(\gamma_2)
\end{cases}
$$

then the scaled $\nu$-gap metric at $\omega_n$ is bounded above by

$$
\delta_{\nu}(W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2
$$
Theorem 2  At a fixed frequency $\omega_n \in \mathbb{I}$, consider the frequency response of a plant model $L(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$, and consider positive definite hermitian matrices $X_i$ and $X_j$ with $X_i = Y_i^{-1} = W_i^*W_i$ and $X_j = Y_j^{-1} = W_j^*W_j$. If a solution exists with achieved objective value for the LMI system:

$$
\begin{bmatrix}
\gamma_2^2 (X_i + C_jY_o C_j^*) & (C_j - C_i) \\
(C_j - C_i)^* & (X_o + C_i^*Y_i C_i)
\end{bmatrix}
(j\omega_n) > 0
$$

$$
\begin{bmatrix}
X_i & I \\
I & Y_i
\end{bmatrix} > 0
$$

$$
\begin{bmatrix}
X_o & I \\
I & Y_o
\end{bmatrix} > 0
$$

with following properties,

$$
\begin{cases}
(X_i^{-1}, X_o) \in \Phi(\gamma_1) \\
(X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2)
\end{cases}
$$

or

$$
\begin{cases}
(Y_i, Y_i^{-1}) \in \Phi(\gamma_1) \\
(Y_i^{-1}, Y_i^{-1}, Y_o, Y_o^{-1}) \in \Psi(\gamma_2)
\end{cases}
$$

then the scaled $\nu$-gap metric at $\frac{\nu}{\omega_n}$ is bounded above by $
\delta_{\nu}(W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1})(j\omega_n) < \gamma_2$.
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

Theorem 3  At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_o$ with $X_i = Y_i^{-1}W_iW_i^*$ and $X_o = Y_o^{-1}W_oW_o^*$. If a solution $(X_i,Y_i,X_o,Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs, 

$$
\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P},C_i)(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P,C_i)(j\omega_n) > 0,
$$

$$
\gamma_2^2 (X_i + C_jY_oC_j^*) (C_j - C_i) (C_j - C_i)^* (X_o + C_i^*Y_iC_i^*) (j\omega_n) > 0
$$

$$
\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} > 0
$$

with following properties,

$$
\left\{ \begin{array}{l}
(X_i^{-1},X_o) \in \Phi(\gamma_1) \\
(X_i,X_i^{-1},X_o,X_o^{-1}) \in \Psi(\gamma_2)
\end{array} \right\} \text{ or } \left\{ \begin{array}{l}
(Y_i,Y_o^{-1}) \in \Phi(\gamma_1) \\
(Y_i^{-1},Y_i,Y_o^{-1},Y_o) \in \Psi(\gamma_2)
\end{array} \right\}
$$

Then the scaled $b_{\hat{P},C_j}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$

$$
b_{\hat{P}_i \hat{P}_o,W_i^{-1}C_jW_o^{-1}}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_\nu(W_i^{-1}C_iW_o^{-1},W_i^{-1}C_jW_o^{-1})(j\omega_n) < \gamma_2
$$
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

**Theorem 3** At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $\tilde{W}_o$ with $X_i = Y_i^{-1} = W_i^*W_i$ and $X_o = Y_o^{-1} = W_o^*W_o$. If a solution $(X_i,Y_i,X_o,Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$
\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P},C_i)^*(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P,C_i)(j\omega_n) > 0,
$$

$$
\begin{bmatrix}
\gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\
(C_j - C_i)^* & (X_o + C_i^* Y_i C_i)
\end{bmatrix} (j\omega_n) > 0
$$

$$
\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} > 0
$$

with following properties,

$$
\begin{cases}
(X_i^{-1},X_o) \in \Phi(\gamma_1) \\
(X_i,X_i^{-1},X_o,X_o^{-1}) \in \Psi(\gamma_2) \quad \text{or} \quad (Y_i,Y_o^{-1}) \in \Phi(\gamma_1)
\end{cases}
$$

Then the scaled $b_{P,C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$

$$
b_{\hat{P}_i} \hat{W}_o, W_i^{-1} C_j W_o^{-1}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_\nu(W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2
$$
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

Theorem 3  At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_0$ with $X_i = Y^{-1}_i = W_iW_i^*$ and $X_o = Y^{-1}_o = W_0W_0^*$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P}, C_i)(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P, C_i)(j\omega_n) > 0,$$

$$\begin{bmatrix} \gamma_2^2(X_i + C_jY_oC_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^*Y_iC_i) \end{bmatrix}(j\omega_n) > 0$$

with following properties,

$$\begin{cases} (X_i^{-1}, X_o) \in \Phi(\gamma_1) \\ (X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2) \end{cases} \quad \text{or} \quad \begin{cases} (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\ (Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2) \end{cases}$$

Then the scaled $b_{\hat{P},C}$ at $\omega_n$ is bounded below by $\gamma_1$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$.

$$b_{W_i \hat{P}W_o, W_i^{-1}C_jW_o^{-1}}(j\omega_n) > \gamma_1^{-1} \quad \text{and} \quad \delta_{\nu}(W_i^{-1}C_iW_o^{-1}, W_i^{-1}C_jW_o^{-1})(j\omega_n) < \gamma_2$$
Simultaneous Scaling of $b_{P,C}$ and $\nu$-gap

**Theorem 3**  
At a fixed frequency $\omega \in \Omega$, consider the frequency response of a plant model $\hat{P}(j\omega_n)$, stabilizing controllers $C_i(j\omega_n)$ and $C_j(j\omega_n)$ and consider positive definite hermitian matrices $W_i$ and $W_o$ with $X_i = Y_i^{-1} = W_iW_i^*$ and $X_o = Y_o^{-1} = W_oW_o^*$. If a solution $(X_i, Y_i, X_o, Y_o)$ exists with achieved objective value $\gamma_1$ and $\gamma_2$ for the LMIs,

$$\gamma_1^2 \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} - T(\hat{P},C_i)^*(j\omega_n) \begin{bmatrix} X_o & 0 \\ 0 & Y_i \end{bmatrix} T(P,C_i)(j\omega_n) > 0,$$

$$\begin{bmatrix} \gamma_2^2 (X_i + C_j Y_o C_j^*) & (C_j - C_i) \\ (C_j - C_i)^* & (X_o + C_i^* Y_i C_i) \end{bmatrix} (j\omega_n) > 0$$

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0$$

$$\begin{bmatrix} X_o & I \\ I & Y_o \end{bmatrix} > 0$$

with following properties,

$$\begin{cases} (X_i^{-1}, X_o) \in \Phi(\gamma_1) \\ (X_i, X_i^{-1}, X_o, X_o^{-1}) \in \Psi(\gamma_2) \end{cases} or \begin{cases} (Y_i, Y_o^{-1}) \in \Phi(\gamma_1) \\ (Y_i^{-1}, Y_i, Y_o^{-1}, Y_o) \in \Psi(\gamma_2) \end{cases}$$

Then the scaled $b_{\hat{P},C}$ at $\omega_n$ is bounded below by $\gamma_1^{-1}$ and the scaled $\nu$-gap metric at $\omega_n$ is bounded above by $\gamma_2$.

$$b_{\hat{P}, \hat{W}, \hat{W}, C, W_o^{-1}}(j\omega_n) > \gamma_1^{-1} \text{ and } \delta_\nu(W_i^{-1} C_i W_o^{-1}, W_i^{-1} C_j W_o^{-1})(j\omega_n) < \gamma_2$$
Simultaneous Scaling Algorithm

Step 1. Chose parameters $0 < \theta_\lambda < 1$ and $0 < \theta_\gamma < 1$

Step 2. Let the initial values, $\gamma_2 > 0$ and $\gamma_1 > 0$, be sufficiently large, and find initial values for $X_{i1}, Y_{i1}, X_{o1}$ and $Y_{o1}$ such that,

$$(Y_{i1}, X_{o1}) \in \Phi(\gamma_1)$$

and  $$(X_{i1}, Y_{i1}, X_{o1}, Y_{o1}) \in \Psi(\gamma_2).$$

Then initialize $k = 1$ and choose $\alpha_1$ and $\beta_1$ such that,

$$\alpha_1 > \lambda_{\text{max}}(X_{i1}, Y_{i1}) \quad \text{and} \quad \beta_1 > \lambda_{\text{max}}(X_{o1}, Y_{o1})$$

Step 3. Compute the analytic centers,

$$(X_{ik}, X_{ok}) =$$

$$\text{ac}\{I < Y_{ik}^{1/2}X_{ik}^{1/2} < \alpha_k I, \ I < Y_{ok}^{1/2}X_{o1}^{1/2} < \beta_k I, \}$$

$$X_{i} \in \Phi(\gamma_{1k}), \text{ and } (X_{i}, X_{o}) \in \Psi(\gamma_{2k})$$

$$(Y_{ik+1}, Y_{ok+1}) =$$

$$\text{ac}\{I < X_{ik}^{1/2}Y_{ik}^{1/2} < \alpha_k I, \ I < X_{ok}^{1/2}Y_{o1}^{1/2} < \beta_k I, \}$$

$$Y_{i} \in \Phi(\gamma_{1k}), \text{ and } (Y_{i}, Y_{o}) \in \Psi(\gamma_{2k})$$

Step 4. If $(X_{ik}, X_{ok})$ do not make scaled $C_j(s)$ and scaled $C_j(s)$ violate the WNC and satisfy the followings,

$$(X_{ik}^{-1}, X_{ok}) \in \Phi(\gamma_1)$$

and  $$(X_{ik}, X_{ik}^{-1}, X_{ok}, X_{ok}^{-1}) \in \Psi(\gamma_2).$$

or if $(Y_{ik}, Y_{ok})$ do not make scaled $C_j(s)$ and scaled $C_j(s)$ violate the WNC and satisfy the followings,

$$(Y_{ik}, Y_{ok}^{-1}) \in \Phi(\gamma_1)$$

and  $$(Y_{ik}^{-1}, Y_{ik}, Y_{ok}^{-1}, Y_{ok}) \in \Psi(\gamma_2).$$

then

$$\gamma_{1k+1} = (1 - \theta_\gamma) \Omega_{\psi}(X_{ik}, Y_{ik}) + \theta_\gamma \gamma_{1k} \quad \text{and} \quad$$

$$\gamma_{2k+1} = (1 - \theta_\gamma) \Omega_{\psi}(X_{ik}, Y_{ik}, X_{ok}, Y_{ok}) + \theta_\gamma \gamma_{2k}$$

where

$$\Omega_{\psi}(X_{ok}, Y_{ik}) = \min \{ \gamma_1 : \Phi(\gamma_1) \geq 0 \} \quad \text{and} \quad$$

$$\Omega_{\psi}(X_{ik}, Y_{ik}, X_{ok}, Y_{ok}) = \min \{ \gamma_2 : \Psi(\gamma_2) \geq 0 \}.$$

Otherwise, $\gamma_{1k+1} = \min \{ \gamma_1 \}$ and $\gamma_{2k+1} = \min \{ \gamma_2 \}$.

$$\alpha_{k+1} = (1 - \theta_\lambda) \lambda_{\text{max}}(X_{ik}, Y_{ik}) + \theta_\lambda \alpha_k$$

$$\beta_{k+1} = (1 - \theta_\lambda) \lambda_{\text{max}}(X_{ok}, Y_{ok}) + \theta_\lambda \beta_k.$$  

Step 5. Stop, if $\gamma_{1k+1} < \varepsilon$ and $\gamma_{1k} < \varepsilon$

and $\gamma_{1k+1} > \varepsilon$ and $\gamma_{1k} > \varepsilon$

Otherwise, $k = k + 1$ and go to Step 3.
Numerical Example

- **Plant model,**
  \[
  A_p = \begin{bmatrix}
  -1 & 0 & 0 & 0 \\
  0 & -0.5 & 0 & 0 \\
  0 & 0 & -3 & 0 \\
  0 & 0 & 0 & -5 \\
  \end{bmatrix},
  B_p = \begin{bmatrix}
  1 \\
  2 \\
  0 \\
  0 \\
  \end{bmatrix},
  C_p = \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1.5 & 0 & 1 \\
  0 & 1 \\
  \end{bmatrix},
  D_p = \begin{bmatrix}
  0 & 0 \\
  \end{bmatrix}
  \]

- **\( \hat{P} \) - stabilizing controller** \( C_0(s) \)
  \[
  A_{c_0} = \begin{bmatrix}
  -1 & 0 \\
  0 & -1 \\
  \end{bmatrix},
  B_{c_0} = \begin{bmatrix}
  2 \\
  0 \\
  \end{bmatrix},
  C_{c_0} = \begin{bmatrix}
  0.5 & 0.5 \\
  \end{bmatrix},
  D_{c_0} = \begin{bmatrix}
  0 & 0 \\
  \end{bmatrix}
  \]

- **\( \hat{P} \) - stabilizing controller** \( C_1(s) \)
  \[
  A_{c_0} = \begin{bmatrix}
  -1 & 0 \\
  0 & -1 \\
  \end{bmatrix},
  B_{c_0} = \begin{bmatrix}
  1 \\
  0 \\
  \end{bmatrix},
  C_{c_0} = \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
  \end{bmatrix},
  D_{c_0} = \begin{bmatrix}
  0.5 & 0 \\
  0 & 0.25 \\
  \end{bmatrix}
  \]

- \[ b_{P_{\text{con}}} = 0.5495 \] and we want to make larger the guaranteed lower bound
Numerical Example

<table>
<thead>
<tr>
<th>Scaling</th>
<th>$b_{\hat{P},C_0}$</th>
<th>$\delta_{v}(C_0, C_1)$</th>
<th>$b_{\hat{P},C_1} \left( &gt; \sin(\arcsin b_{\hat{P},C_0} - \arcsin \delta_{v}(C_0, C_1)) \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without</td>
<td>0.5495</td>
<td>0.5407</td>
<td>&gt; 0.0105</td>
</tr>
<tr>
<td>$b_{\hat{P},C_0}^{-1}$ only</td>
<td>0.9964</td>
<td>0.9921</td>
<td>&gt; 0.0409</td>
</tr>
<tr>
<td>$b_{\hat{P},C_0}^{-1}$ and $\delta_{v}(C_0, C_1)$</td>
<td>0.6362</td>
<td>0.4521</td>
<td>&gt; 0.2188</td>
</tr>
</tbody>
</table>

- **Scalings for maximizing**

\[
W_o = \begin{bmatrix}
1.3159 & 1.0086 - 0.0016i \\
1.0086 + 0.0016i & 1.3073
\end{bmatrix}, \quad W_i = \begin{bmatrix}
0.2088 & 0.07 - 0.0025i \\
0.07 + 0.0025i & 0.3388
\end{bmatrix}
\]

- **Simultaneous scalings increased the generalized stability margin and decreased the $\delta_{v}$-gap**